

Quantum entanglement is a unique state of quantum superposition.

It has been studied mainly from a scientific interest as an evidence of quantum mechanics.

Recently, it is also being studied as a basic element for quantum information processing such as quantum computing and quantum communication.

This chapter describes quantum entanglement.

Review of polarization state of a photon

The polarization state of a photon is a good material for explaining quantum entanglement.

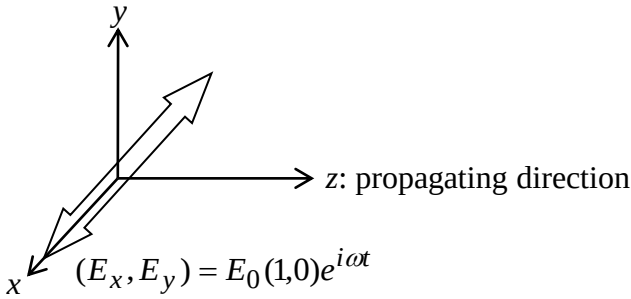
Classically, light is an electromagnetic wave.

Electric/magnetic field oscillates along the transversal direction while propagating.

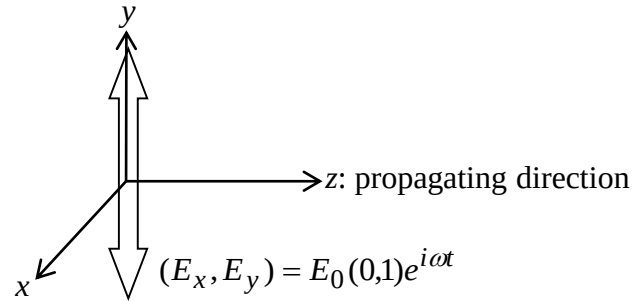
Since the transversal direction is two-dimensional, the direction of the oscillation has two degrees of freedom.

The way of the oscillation in the transversal two-dimensional plane is called “polarization state”.

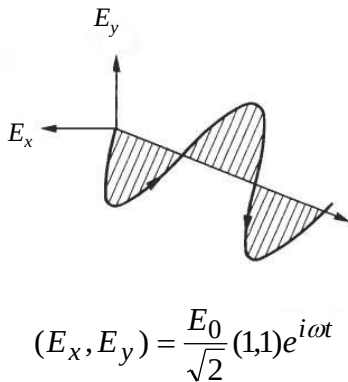
horizontal linear polarization



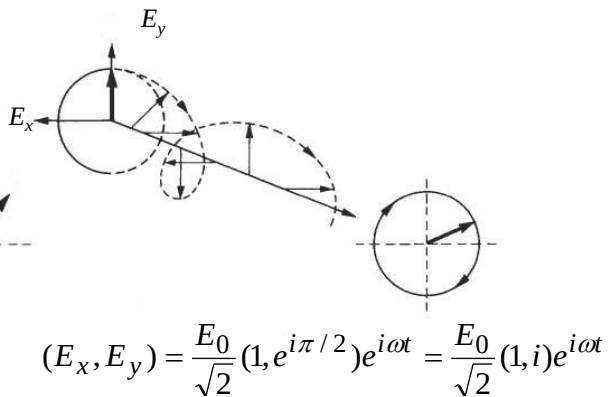
vertical linear polarization



45deg. right diagonal



right circular



45deg. left diagonal

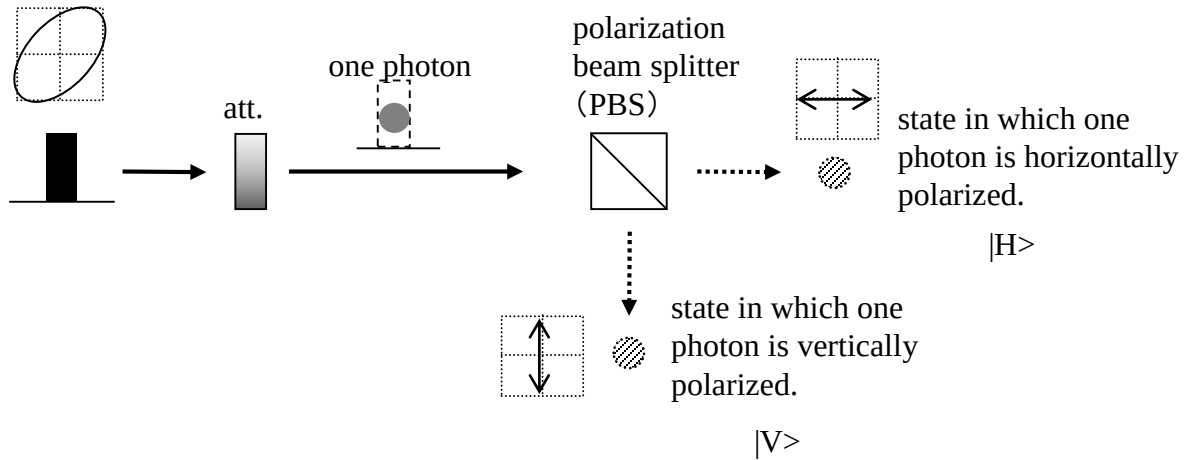
$$(E_x, E_y) = \frac{E_0}{\sqrt{2}}(1, e^{i\pi})e^{i\omega t} = \frac{E_0}{\sqrt{2}}(1, -1)e^{i\omega t}$$

left circular

$$(E_x, E_y) = \frac{E_0}{\sqrt{2}}(1, e^{-i\pi/2})e^{i\omega t} = \frac{E_0}{\sqrt{2}}(1, -i)e^{i\omega t}$$

The above description is for classical lightwave.

How about the polarization state of a photon ?



$|H\rangle$ and $|V\rangle$ are eigenstates of the physical quantity operator of "whether a photon is horizontal or vertical". Thus, an arbitrary photon state is a superposition of a horizontal photon and a vertical photon, i.e., a linear combination of $|H\rangle$ and $|V\rangle$, where $|H\rangle$ and $|V\rangle$ are the bases forming a Hilbert space.

$$|\Psi\rangle = c_H |H\rangle + c_V |V\rangle$$

For example,

45deg. right diagonal: $|D\rangle = \frac{1}{\sqrt{2}} |H\rangle + \frac{1}{\sqrt{2}} |V\rangle$

45deg. left diagonal: $|\bar{D}\rangle = \frac{1}{\sqrt{2}} |H\rangle - \frac{1}{\sqrt{2}} |V\rangle$

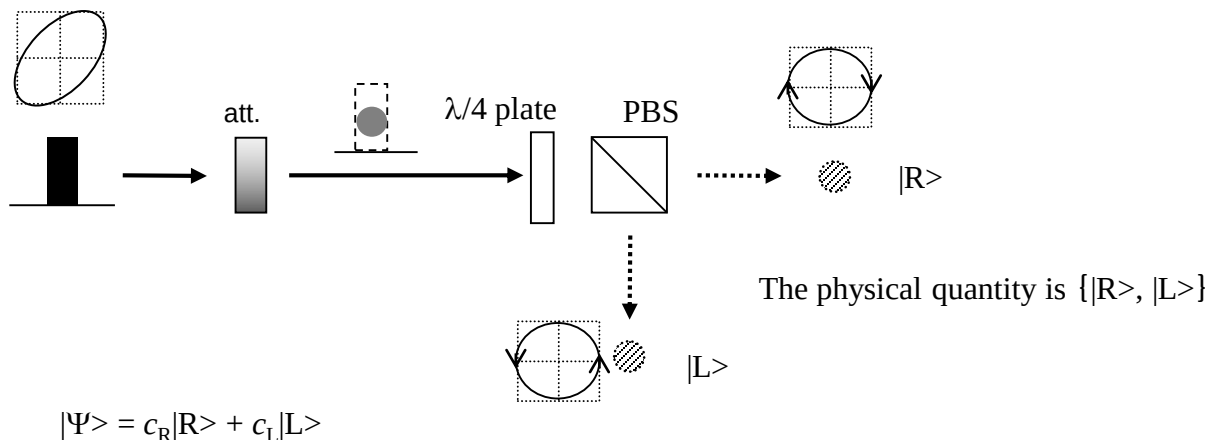
right circular: $|R\rangle = \frac{1}{\sqrt{2}} |H\rangle + \frac{i}{\sqrt{2}} |V\rangle$

left circular: $|L\rangle = \frac{1}{\sqrt{2}} |H\rangle - \frac{i}{\sqrt{2}} |V\rangle$

Generally speaking, bases are eigenstates of a physical quantity operator.

Another physical quantity operator has another set of eigenstates (bases),

and then a quantum state can be expressed by the other bases, depending on the physical quantity.



Suppose that two photons are regarded as one quantum state (i.e., a product state).

For a two-dimensional Hilbert space, a product state is generally expressed as

$$\begin{aligned}
 |\Psi\rangle &= |\Psi\rangle_1 \otimes |\Psi\rangle_2 \\
 &= \{c_{1H}|H\rangle_1 + c_{1V}|V\rangle_1\} \otimes \{c_{2H}|H\rangle_2 + c_{2V}|V\rangle_2\} \\
 &= c_{11}|H\rangle_1|H\rangle_2 + c_{12}|H\rangle_1|V\rangle_2 \\
 &\quad + c_{21}|V\rangle_1|H\rangle_2 + c_{22}|V\rangle_1|V\rangle_2
 \end{aligned}$$

$|H\rangle_1$: photon #1 is state $|H\rangle$
 $|V\rangle_1$: photon #1 is state $|V\rangle$
 $|H\rangle_2$: photon #2 is state $|H\rangle$
 $|V\rangle_2$: photon #2 is state $|V\rangle$

However, a particular product state expressed as below can exist.

$$\begin{aligned}
 |\Psi_1\rangle &= \frac{1}{\sqrt{2}}\{|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2\} & |\Psi_2\rangle &= \frac{1}{\sqrt{2}}\{|H\rangle_1|H\rangle_2 - |V\rangle_1|V\rangle_2\} \\
 |\Psi_3\rangle &= \frac{1}{\sqrt{2}}\{|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2\} & |\Psi_4\rangle &= \frac{1}{\sqrt{2}}\{|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2\}
 \end{aligned}$$

Such a state is called “*quantum entanglement*” or “*quantum entangled state*”, which has unique properties much different from the classical sense.

Properties of quantum entanglement

When the above state is observed, the resultant state is $|H\rangle_1|H\rangle_2$ or $|V\rangle_1|V\rangle_2$ (in case of $|\Psi_1\rangle$).

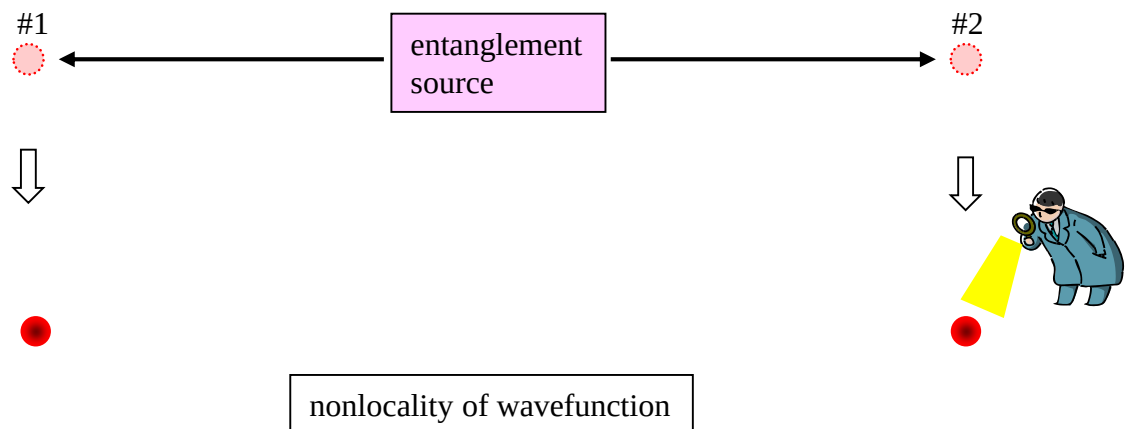
- The observed state, $|H\rangle$ or $|V\rangle$, is random for each photon.
- There is a correlation between the observed states of each photon.

When photon #1 is $|H\rangle$, photon #2 is always $|H\rangle$.

When photon #1 is $|V\rangle$, photon #2 is always $|V\rangle$.

- This correlation is irrespective of the distance between the two photons.

The state of photon #1 is determined instantly at a moment when photon #2 is observed.



- A correlation is made even for different bases.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2\}$$

$$\left(\begin{array}{l} |R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle) \\ |L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle) \end{array} \right) \rightarrow \left(\begin{array}{l} |H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \\ |V\rangle = -\frac{i}{\sqrt{2}}(|R\rangle - |L\rangle) \end{array} \right)$$

$$= \frac{1}{2\sqrt{2}}\{(|R\rangle_1 + |L\rangle_1)(|R\rangle_2 + |L\rangle_2) - (|R\rangle_1 - |L\rangle_1)(|R\rangle_2 - |L\rangle_2)\}$$

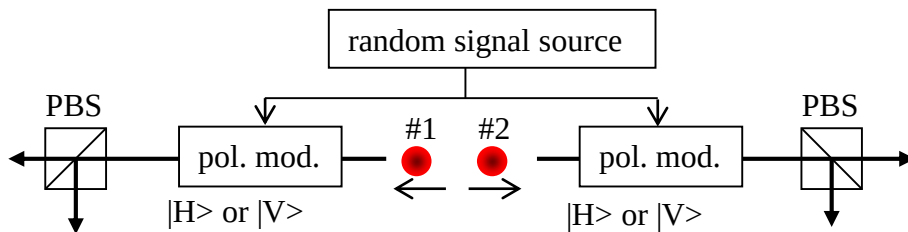
$$= \frac{1}{2\sqrt{2}}\{|R\rangle_1|R\rangle_2 + |R\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2 + |L\rangle_1|L\rangle_2$$

$$- |R\rangle_1|R\rangle_2 + |R\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2 - |L\rangle_1|L\rangle_2\}$$

$$= \frac{1}{\sqrt{2}}\{|R\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2\}$$

Difference from classical correlation

A correlation looks similar to the above can be classically obtained by a system below.



The generated state is

$$|\Psi\rangle = |H\rangle_1|H\rangle_2 \quad \text{or} \quad |\Psi\rangle = |V\rangle_1|V\rangle_2$$

- $|H\rangle$ or $|V\rangle$ is random for each photon.
- The observed states of each photon have a correlation.

When photon #1 is $|H\rangle$, photon #2 is always $|H\rangle$.

When photon #1 is $|V\rangle$, photon #2 is always $|V\rangle$.

looks like entanglement
at a first glance

But, in different bases,,,,,,

$$|\Psi\rangle = \frac{1}{2}\{|R\rangle_1|R\rangle_2 + |R\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2 + |L\rangle_1|L\rangle_2\}$$

or

no correlation

$$|\Psi\rangle = -\frac{1}{2}\{|R\rangle_1|R\rangle_2 - |R\rangle_1|L\rangle_2 - |L\rangle_1|R\rangle_2 + |L\rangle_1|L\rangle_2\}$$

no correlation

A correlation is made irrespective of basis in *quantum* entanglement.

A state below looks like quantum entanglement.

$$|\Psi\rangle = c_1 |H\rangle_1 |H\rangle_2 + c_2 |V\rangle_1 |V\rangle_2$$

The observed state is random for one photon.
When photon #1 is $|H\rangle$, photon #2 is always $|H\rangle$.
When photon #1 is $|V\rangle$, photon #2 is always $|V\rangle$.

But, in another bases,,,,,

$$\begin{aligned} |\Psi\rangle &= \frac{1}{2} \{c_1 (|R\rangle_1 + |L\rangle_1) (|R\rangle_2 + |L\rangle_2) - c_2 (|R\rangle_1 - |L\rangle_1) (|R\rangle_2 - |L\rangle_2)\} \\ &= \frac{1}{2} \{(c_1 - c_2) |R\rangle_1 |R\rangle_2 + (c_1 + c_2) |R\rangle_1 |L\rangle_2 \\ &\quad + (c_1 + c_2) |L\rangle_1 |R\rangle_2 + (c_1 - c_2) |L\rangle_1 |L\rangle_2\} \end{aligned}$$

No correlation for arbitrary c_1 and c_2 . $\{c_1, c_2\}$ should satisfy a particular condition for entanglement.
For a correlation to be made in $\{|L\rangle, |R\rangle\}$ basis,

$$\begin{aligned} c_1 = c_2 &\rightarrow |\Psi\rangle = c_1 \{ |R\rangle_1 |L\rangle_2 + |L\rangle_1 |R\rangle_2 \} \xrightarrow{\text{(normalized)}} |\Psi\rangle = \frac{1}{\sqrt{2}} \{ |R\rangle_1 |L\rangle_2 + |L\rangle_1 |R\rangle_2 \} \\ \text{or} \\ c_1 = -c_2 &\rightarrow |\Psi\rangle = c_1 \{ |R\rangle_1 |R\rangle_2 + |L\rangle_1 |L\rangle_2 \} \xrightarrow{\text{(normalized)}} |\Psi\rangle = \frac{1}{\sqrt{2}} \{ |R\rangle_1 |R\rangle_2 + |L\rangle_1 |L\rangle_2 \} \end{aligned}$$



Quantum-entangled state is $|\Psi\rangle = \frac{1}{\sqrt{2}} \{ |H\rangle_1 |H\rangle_2 \pm |V\rangle_1 |V\rangle_2 \}$

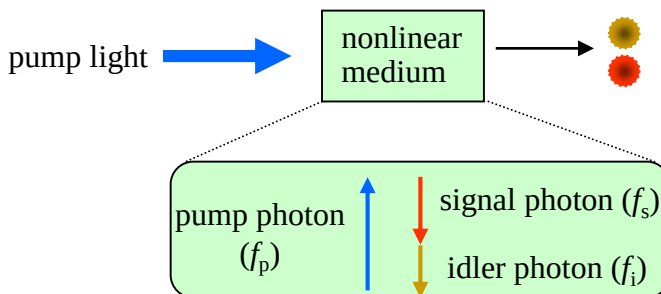
Similarly, $|\Psi\rangle = \frac{1}{\sqrt{2}} \{ |H\rangle_1 |V\rangle_2 \pm |V\rangle_1 |H\rangle_2 \}$ is also quantum entanglement.

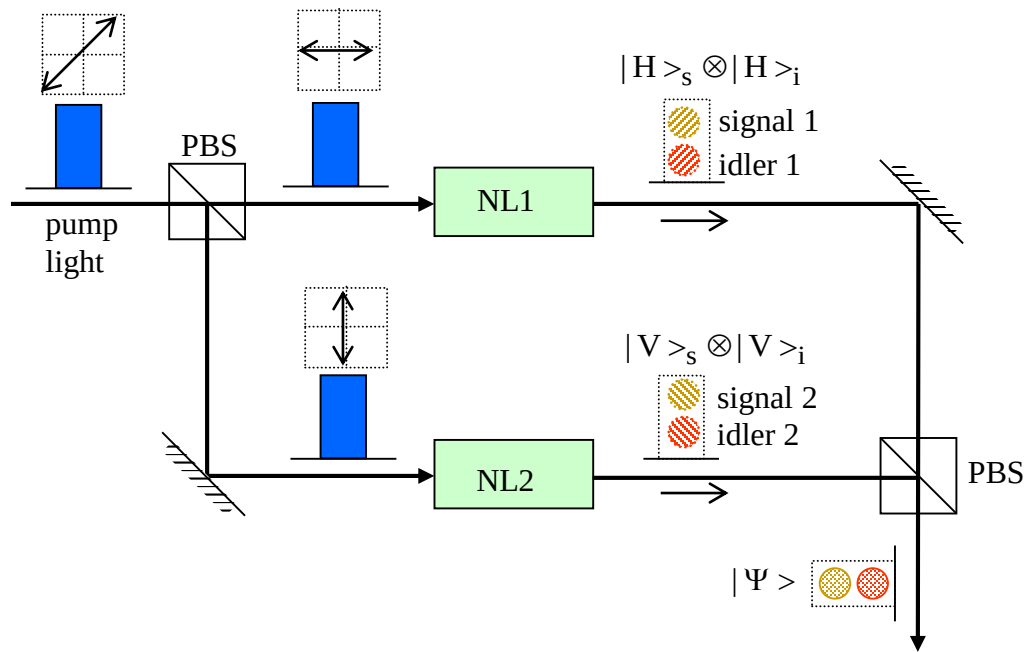
Generally speaking, perfect quantum entanglement is

$$\left\{ \begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2}} \{ |\phi_1\rangle_a |\phi_1\rangle_b \pm |\phi_2\rangle_a |\phi_2\rangle_b \} \\ |\Psi\rangle &= \frac{1}{\sqrt{2}} \{ |\phi_1\rangle_a |\phi_2\rangle_b \pm |\phi_2\rangle_a |\phi_1\rangle_b \} \end{aligned} \right. \quad \left\{ |\phi_1\rangle, |\phi_2\rangle \right\}: \text{(two-dimensional) basis}$$

How to generate quantum entanglement

When intense pump light is injected into a nonlinear medium, signal and idler photons are simultaneously generated from a pump photon via nonlinear parametric interaction.





Suppose that the probability of one-pair generation = r .

Then,

the probability that both NLs generate a photon pair = r^2 ,

the probability that either one of NLs generates a photon pair = $r(1 - r)$.

$$\downarrow r \ll 1$$

$r(1 - r) \gg r^2$: A photon pair is generated either at NL1 or NL2.

When a photon pair is output, we do not know which NL generates it. $\rightarrow$ superpositioned state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \{ |H\rangle_s |H\rangle_i + e^{i\phi} |V\rangle_s |V\rangle_i \} \quad (\phi: \text{phase difference through the two paths})$$

$\nearrow$ normalization $\uparrow$ from NL1 $\uparrow$ from NL2

$$\Downarrow \phi = 0$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \{ |H\rangle_s |H\rangle_i + |V\rangle_s |V\rangle_i \} \quad \text{entanglement}$$

Applications of quantum entanglement

Quantum cryptography

Quantum Computing

Quantum teleportation

Quantum repeater

Quantum networks

etc.