

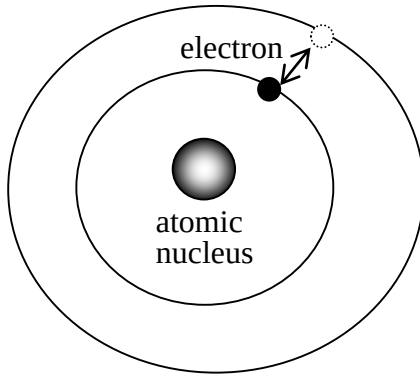
V. Light amplification & Spontaneous emission

1

Lasers are based on spontaneous emission and light amplification, which are kinds of quantum phenomenon. This chapter quantum-mechanically describes light amplification.

Energy level of a material system

A matter is composed of atoms, and an atom is composed of a nucleus and electrons.



Possible orbits of electrons are discrete.



The energy of an atom is discrete.

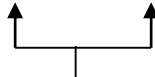
{ distant orbit: The potential energy is high.
close orbit: The potential energy is low.

In quantum-mechanical terms, eigenvalues of the energy operator of an atom are discrete.

energy eigenvalue (discrete)



$$\hat{H} |\phi_i\rangle = E_i |\phi_i\rangle$$



eigenstate corresponding to an electron orbit.

(schematic image of energy states)

energy ↑

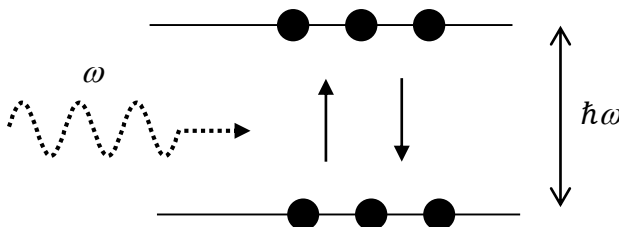
high energy state

low energy state

(representing each atom)

Interaction between light and a matter

When lightwave is injected into a matter system under the condition of the photon energy being equal to the energy difference between two energy-levels, as illustrated below, interaction (or energy exchange) between light and the matter efficiently occurs.



Transition from a low energy level to a high energy level $\iff$ light absorption

a high energy level to a low energy level $\iff$ light emission

There are two approaches to describe the above light-atom interaction;

Semi-classical theory: an atom is quantum, and light is classical.

Full-quantum theory: an atom and light are both quantum.

The semi-classical theory is convenient to intuitively understand the resonant interaction.

The full-quantum theory exactly describes the interaction, from which spontaneous emission is derived.

Behavior of an atom is the main concern in the semi-classical theory

The atom's state changes while interacting with light.

In quantum mechanics, the state evolution is described by Schrödinger equation.

We will see how an atom temporally evolves through the interaction, using Schrödinger equation.

Schrödinger equation:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

$\hat{H}$: energy operator of an atom (Hamiltonian)

$|\Psi(t)\rangle$: quantum state of an atom

First, we express the atom's state as a superposition of the atom's energy eigenstates:

$$|\Psi(t)\rangle = \sum_j c_j(t) e^{-iE_j t / \hbar} |\phi_j\rangle$$

$\uparrow$
 probability amplitude possibly varying through the interaction

$\longleftarrow$ phase evolution without the interaction (Chap. 2)

where $|\phi_j\rangle$ is an eigenstate of the energy operator of the atom $\hat{H}_0$.

$$\hat{H}_0 |\phi_j\rangle = E_j |\phi_j\rangle \quad (E_j : \text{energy eigenvalue})$$

The total energy operator, when the atom interacts with light, includes the interaction energy operator in addition.

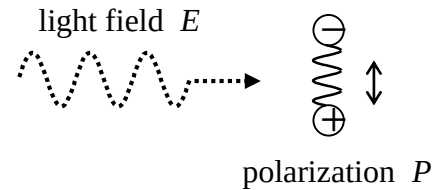
$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$\hat{H}_{\text{int}}$ is the energy operator for the atom-light interaction, which is derived below.

We assume the dipole interaction, whose classical energy is

$$V_{\text{int}} = -PE = e \sum_j r_j E = eDE$$

$$\left(\begin{array}{ll} P = -e \sum_j r_j & D \equiv \sum_j r_j \\ e: \text{electron charge} & \\ r_j: \text{position of the } j\text{-th electron} & \end{array} \right)$$



The interaction energy operator is obtained by replacing D with an operator, while the light is classical in the semi-classical theory.

$$\hat{H}_{\text{int}} = e\hat{D}(E_0 e^{-i\omega t} + E_0^* e^{i\omega t})$$

$$\begin{aligned}
i\hbar \frac{d}{dt} \sum_j c_j(t) e^{-iE_j t/\hbar} |\phi_j\rangle &= (\hat{H}_0 + \hat{H}_{\text{int}}) \sum_j c_j(t) e^{-iE_j t/\hbar} |\phi_j\rangle \\
i\hbar \sum_j \left(\frac{dc_j}{dt} - i \frac{E_j}{\hbar} \right) e^{-iE_j t/\hbar} |\phi_j\rangle &= \sum_j \{ c_j e^{-iE_j t/\hbar} E_j |\phi_j\rangle + \hat{H}_{\text{int}} c_j e^{-iE_j t/\hbar} |\phi_j\rangle \} \\
i\hbar \sum_j \frac{dc_j}{dt} e^{-iE_j t/\hbar} |\phi_j\rangle &= \sum_j \hat{H}_{\text{int}} c_j e^{-iE_j t/\hbar} |\phi_j\rangle
\end{aligned}$$

Take the inner product with $|\phi_k\rangle$

$$\begin{aligned}
i\hbar \frac{dc_k}{dt} e^{-iE_k t/\hbar} &= \sum_j c_j e^{-iE_j t/\hbar} \langle \phi_k | \hat{H}_{\text{int}} | \phi_j \rangle & \left(\omega_{kj} \equiv \frac{E_k - E_j}{\hbar} \right) \\
\frac{dc_k}{dt} &= -\frac{i}{\hbar} \sum_j c_j \langle \phi_k | \hat{H}_{\text{int}} | \phi_j \rangle e^{i\omega_{kj} t} \\
&= -\frac{i}{\hbar} e(E_0 e^{-i\omega t} + E_0^* e^{i\omega t}) \sum_j c_j D_{kj} e^{i\omega_{kj} t} & \left(D_{kj} \equiv \langle \phi_k | \hat{D} | \phi_j \rangle \right)
\end{aligned}$$

Resonant transition

The above simultaneous equations cannot be exactly solved.

Here, an approximate approach is employed to solve them, where c_j in the right-hand side is assumed to be constant in a short time.

$$\frac{dc_k}{dt} = -\frac{i}{\hbar} e(E_0 e^{-i\omega t} + E_0^* e^{i\omega t}) \sum_j c_j(0) D_{kj} e^{i\omega_{kj} t}$$

Here, we assume that the atom is initially at one of the eigenstate $|\phi_i\rangle$, not a superposition state.

$$\begin{cases} c_i^{(0)} = 1 \\ c_j^{(0)} = 0 \end{cases} \quad \text{for } j \neq i$$

Then, the above equation can be solved as

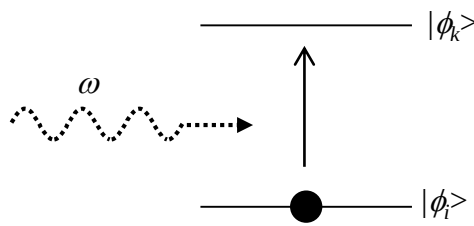
$$\begin{aligned}
\frac{dc_k}{dt} &= -i \frac{e}{\hbar} D_{ki} (E_0 e^{-i\omega t} + E_0^* e^{i\omega t}) e^{i\omega_{ki} t} = -i \frac{e}{\hbar} D_{ki} \{ E_0 e^{i(\omega_{ki} - \omega)t} + E_0^* e^{i(\omega_{ki} + \omega)t} \} \\
&\downarrow \\
c_k(t) &= -\frac{e}{\hbar} D_{ki} \left\{ E_0 \frac{e^{i(\omega_{ki} - \omega)t} - 1}{\omega_{ki} - \omega} + E_0^* \frac{e^{i(\omega_{ki} + \omega)t} - 1}{\omega_{ki} + \omega} \right\}
\end{aligned}$$

This expression indicates that $c_k(t)$ is large when $\omega_{ki} \pm \omega \approx 0$.

$$(1) \text{ The first term is large when } \omega_{ki} \approx \omega \rightarrow \frac{E_k - E_i}{\hbar} = \omega \rightarrow E_k = E_i + \hbar\omega$$

The atom transits from the initial state to a higher energy state with an energy difference of $\hbar\omega$, which equals to the photon energy of the incident light.

We are now considering a closed system. Thus, the energy that the atom obtains should come from somewhere because of the energy conservation. The candidate is the light, which is reasonable since the energy lost by the atom equals to the photon energy.

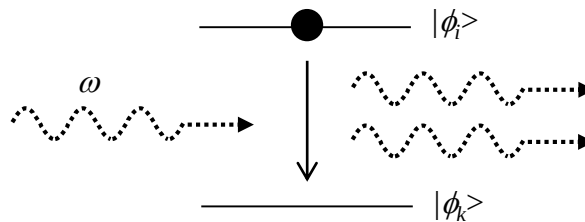


(2) The second term is large when $-\omega_{ki} \approx \omega \rightarrow -\frac{E_k - E_i}{\hbar} = \omega \rightarrow E_k = E_i - \hbar\omega$

The atom transits from the initial state to a lower energy state with an energy difference of $\hbar\omega$, which equals to the photon energy of the incident light.

Consideration similar to case (1) suggests that the atom transits from the initial state to a lower energy state, giving the energy from a photon (maybe).

This consideration suggests that the atom transits from the initial state to a lower energy state while the lost energy turns to be a photon (maybe).



Resonant process: Interaction occurs most efficiently when the photon energy is coincident with an energy difference of an atom.

The probability of the state transition is estimated as follows.

First, we assume $\omega_{ki} \approx -\omega$. Then, (first term) $\ll$ (second term) (i.e., emission)

$$c_k(t) = -\frac{e}{\hbar} D_{ki} E_0 \frac{e^{i(\omega_{ki} + \omega)t} - 1}{\omega_{ki} + \omega}$$

The probability of the state being at the k -th energy state at time t is

$$|c_k|^2 = \frac{e^2}{\hbar^2} |D_{ki}|^2 |E_0|^2 \frac{\sin^2 [(\omega_{ki} + \omega) t / 2] / 4}{(\omega_{ki} + \omega)^2}$$

which is equivalent to the transition probability from the i -th state to the k -th state.

This consideration suggests that the transition probability is proportional to the light intensity.

In summary,

- Transition occurs efficiently between energy states whose energy difference equals to the photon energy.

An atom transits from an upper energy state to a lower one $\Leftrightarrow$ light emission (maybe)

a lower energy state to an upper one $\Leftrightarrow$ light absorption (maybe)

- The transition probability is proportional to the light intensity
(no transition when no light $\rightarrow$ no spontaneous transition)

- The transition probability from the upper to the lower and that from the lower to the upper are equal.

The previous section suggests the resonant interaction, i.e., an atom and light efficiently interact with each other when the photon energy equals to an energy difference of the atom system.

However, there are some unclear issues:

- Photon emission and absorption are suggested by a circumstantial evidence, not by direct derivation.
- Spontaneous emission is not derived.
- Other information related to light amplification are missed, such as
light amplitude, amplitude fluctuation, photon number fluctuation

In order to fully understand light amplification, we need the full-quantum theory, which is presented in this section.

First, the light is quantized as well as atoms in the energy operator.

Based on the semi-classical theory, the interaction energy operator can be expressed as:

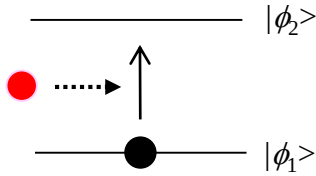
$$\hat{H}_{\text{int}} = i\hbar\kappa \sum_j (\hat{\pi}_j^\dagger \hat{a} - \hat{\pi}_j \hat{a}^\dagger) \quad \begin{array}{l} \text{(The subscript } j \text{ represents the } j\text{-th atom.)} \\ \text{(} \kappa \text{: constant representing interaction efficiency)} \end{array}$$

where

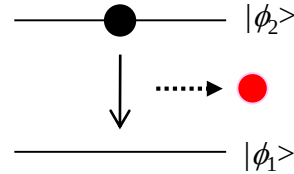
$\hat{\pi} \equiv |\phi_1\rangle\langle\phi_2|$: the atom's transition operator from the upper state $|\phi_2\rangle$ to the lower one $|\phi_1\rangle$

$\hat{\pi}^\dagger \equiv |\phi_2\rangle\langle\phi_1|$: the atom's transition operator from the lower state $|\phi_1\rangle$ to the upper one $|\phi_2\rangle$

$\hat{\pi}_j^\dagger \hat{a}$ { A photon is absorbed.
An atom is excited.



$\hat{\pi}_j \hat{a}^\dagger$ { A photon is emitted.
An atom is relaxed.



Hereafter, we employ the Heisenberg picture.

Heisenberg picture: a time-dependent operator is derived first, and then its expectation value with respect to an initial state is evaluated.

$$\langle \Psi(t) | \hat{A} | \Psi(t) \rangle = \langle \Psi(0) | \underbrace{e^{(i/\hbar)\hat{H}t} \hat{A} e^{-(i/\hbar)\hat{H}t}}_{\hat{A}_H(t)} | \Psi(0) \rangle$$

$\hat{A}_H(t)$: time-dependent Heisenberg operator

$\hat{A}_H(t)$ follows Heisenberg equation of motion:

$$\frac{d}{dt} \hat{A}_H(t) = \frac{i}{\hbar} [\hat{H}, \hat{A}_H(t)] = \frac{i}{\hbar} (\hat{H} \hat{A}_H - \hat{A}_H \hat{H}) \quad \left[\hat{H} : \text{Hamiltonian (energy operator)} \right]$$

The energy in the present system is composed of the original energies and the interaction energy:

$$\hat{H} = \underbrace{\hbar\omega \hat{a}^\dagger \hat{a}}_{\text{(light)}} + \underbrace{\sum_j E_j \hat{\pi}_j^\dagger \hat{\pi}_j}_{\text{(atoms)}} + \underbrace{i\hbar\kappa \sum_j (\hat{\pi}_j^\dagger \hat{a} - \hat{\pi}_j \hat{a}^\dagger)}_{\text{(interaction)}}$$

($\hbar\omega/2$ is neglected because it does not matter here.)

About $\sum_j E_j \hat{\pi}_j^\dagger \hat{\pi}_j$
--

An atom state is a superposition of $|\phi_1\rangle$ and $|\phi_2\rangle$.

$$\left\{ \begin{array}{l} |\Psi\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle \rightarrow c_1 = \langle \phi_1 | \Psi \rangle \quad c_2 = \langle \phi_2 | \Psi \rangle \\ \langle \Psi | = c_1^* \langle \phi_1 | + c_2^* \langle \phi_2 | \rightarrow c_1^* = \langle \Psi | \phi_1 \rangle \quad c_2^* = \langle \Psi | \phi_2 \rangle \end{array} \right. \quad \left(\langle \phi_i | \phi_j \rangle = \delta_{ij} \right)$$



$$\left\{ \begin{array}{l} |\Psi\rangle = \langle \phi_1 | \Psi \rangle |\phi_1\rangle + \langle \phi_2 | \Psi \rangle |\phi_2\rangle \\ \langle \Psi | = \langle \Psi | \phi_1 \rangle \langle \phi_1 | + \langle \Psi | \phi_2 \rangle \langle \phi_2 | \end{array} \right.$$

A quantum state is normalized.

$$\langle \Psi | \Psi \rangle = 1$$

$$\{ \langle \Psi | \phi_1 \rangle \langle \phi_1 | + \langle \Psi | \phi_2 \rangle \langle \phi_2 | \} \{ \langle \phi_1 | \Psi \rangle |\phi_1\rangle + \langle \phi_2 | \Psi \rangle |\phi_2\rangle \} = 1$$

$$\langle \Psi | \phi_1 \rangle \langle \phi_1 | \Psi \rangle + \langle \Psi | \phi_2 \rangle \langle \phi_2 | \Psi \rangle = 1$$

$$\langle \Psi | \{ |\phi_1\rangle \langle \phi_1| + |\phi_2\rangle \langle \phi_2| \} | \Psi \rangle = 1$$

1 (identity operator)

$$\left(\hat{H}_a | \phi_k \rangle = E_k | \phi_k \rangle \right)$$

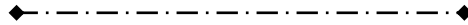
Applying the identity operators to the energy operator.

$$\begin{aligned} \hat{H}_a &= \left\{ \sum_{k'=1,2} | \phi_{k'} \rangle \langle \phi_{k'} | \right\} \hat{H}_a \left\{ \sum_{k=1,2} | \phi_k \rangle \langle \phi_k | \right\} = \left\{ \sum_{k'=1,2} | \phi_{k'} \rangle \langle \phi_{k'} | \right\} \left\{ \sum_{k=1,2} E_k | \phi_k \rangle \langle \phi_k | \right\} \\ &= E_1 | \phi_1 \rangle \langle \phi_1 | + E_2 | \phi_2 \rangle \langle \phi_2 | \end{aligned}$$

We regard E_1 as a reference of the energy level, and assume $E_1 = 0$.

$$\hat{H}_a = \Delta E | \phi_2 \rangle \langle \phi_2 | = \Delta E | \phi_2 \rangle \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | = \Delta E \hat{\pi}^\dagger \hat{\pi} \quad \left(\Delta E = E_2 - E_1 \right)$$

An ensemble of atoms: $\sum_j E_j \hat{\pi}_j^\dagger \hat{\pi}_j$ where $E_j \equiv \Delta E$ for j th atom



Heisenberg equation for light:

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{a}] = \frac{i}{\hbar} \{ [\hbar\omega \hat{a}^\dagger \hat{a}, \hat{a}] + \sum_j [E_j \hat{\pi}_j^\dagger \hat{\pi}_j, \hat{a}] + i\hbar\kappa \sum_j [(\hat{\pi}_j^\dagger \hat{a} - \hat{\pi}_j \hat{a}^\dagger), \hat{a}] \} = -i\omega \hat{a} - \kappa \sum_j \hat{\pi}_j$$

$$\left(\begin{array}{l} [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger \hat{a} \hat{a} - \hat{a} \hat{a}^\dagger \hat{a} = \hat{a}^\dagger \hat{a} \hat{a} - (\hat{a}^\dagger \hat{a} + 1) \hat{a} = -\hat{a} \\ \sum_j [(\hat{\pi}_j^\dagger \hat{a} - \hat{\pi}_j \hat{a}^\dagger), \hat{a}] = \sum_j \{ (\hat{\pi}_j^\dagger \hat{a} - \hat{\pi}_j \hat{a}^\dagger) \hat{a} - \hat{a} (\hat{\pi}_j^\dagger \hat{a} - \hat{\pi}_j \hat{a}^\dagger) \} \\ = \sum_j \{ (\hat{\pi}_j^\dagger \hat{a} \hat{a} - \hat{\pi}_j \hat{a}^\dagger \hat{a}) - (\hat{\pi}_j^\dagger \hat{a} \hat{a} - \hat{\pi}_j \hat{a} \hat{a}^\dagger) \} = \sum_j \{ -\hat{\pi}_j \hat{a}^\dagger \hat{a} + \hat{\pi}_j \hat{a} \hat{a}^\dagger \} \\ = \sum_j \hat{\pi}_j (\hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a}) = \sum_j \hat{\pi}_j \end{array} \right) \quad \left([\hat{a}, \hat{a}^\dagger] = 1 \right)$$

$$\frac{d\hat{\pi}_j}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{\pi}_j] = -\frac{i}{\hbar} E_j \hat{\pi}_j - \kappa (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) \hat{a}$$

$$\left(\begin{array}{l} \sum_k E_k [\hat{\pi}_k^\dagger \hat{\pi}_k, \hat{\pi}_j] = \sum_k E_k (\hat{\pi}_k^\dagger \hat{\pi}_k \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_k^\dagger \hat{\pi}_k) = -E_j \hat{\pi}_j \\ \left(\begin{array}{l} \hat{\pi} \hat{\pi}^\dagger \hat{\pi} = |1\rangle\langle 2| 2\rangle\langle 1| 1\rangle\langle 2| = |1\rangle\langle 2| = \hat{\pi} \\ \hat{\pi}^\dagger \hat{\pi} \hat{\pi} = |2\rangle\langle 1| 1\rangle\langle 2| 1\rangle\langle 2| = 0 \end{array} \right) \\ \sum_k [(\hat{\pi}_k^\dagger \hat{a} - \hat{\pi}_k \hat{a}^\dagger), \hat{\pi}_j] = \sum_k \{(\hat{\pi}_k^\dagger \hat{a} - \hat{\pi}_k \hat{a}^\dagger) \hat{\pi}_j - \hat{\pi}_j (\hat{\pi}_k^\dagger \hat{a} - \hat{\pi}_k \hat{a}^\dagger)\} \\ = \sum_k \{(\hat{\pi}_k^\dagger \hat{\pi}_j \hat{a} - \hat{\pi}_k \hat{\pi}_j \hat{a}^\dagger) - (\hat{\pi}_j \hat{\pi}_k^\dagger \hat{a} - \hat{\pi}_j \hat{\pi}_k \hat{a}^\dagger)\} = (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) \hat{a} \end{array} \right)$$

The above differential equations are simplified by variable transformation as:

$$\hat{a} \rightarrow \hat{a} e^{-i\omega t}$$

$$\hat{\pi}_j \rightarrow \hat{\pi}_j e^{-i(E_j/\hbar)t}$$



$$\frac{d\hat{a}}{dt} = -\kappa \sum_j \hat{\pi}_j e^{-i(E_j/\hbar - \omega)t} \quad \frac{d\hat{\pi}_j}{dt} = -\kappa (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) \hat{a} e^{i(E_j/\hbar - \omega)t}$$

We solve these simultaneous equations by a successive approximation.

$$\left\{ \begin{array}{l} \frac{d\hat{a}^{(1)}}{dt} = -\kappa \sum_j \hat{\pi}_j^{(0)} e^{-i(E_j/\hbar - \omega)t} \\ \frac{d\hat{\pi}_j^{(1)}}{dt} = -\kappa (\hat{\pi}_j^{(0)\dagger} \hat{\pi}_j^{(0)} - \hat{\pi}_j^{(0)} \hat{\pi}_j^{(0)\dagger}) \hat{a}^{(0)} e^{i(E_j/\hbar - \omega)t} \end{array} \right. \quad \left(\begin{array}{l} \hat{a}^{(0)}, \hat{\pi}_j^{(0)} : \text{initial states} \\ \hat{a}^{(1)}, \hat{\pi}_j^{(1)} : \text{first-order solutions} \end{array} \right)$$



$$\left\{ \begin{array}{l} \hat{a}^{(1)} = \hat{a}^{(0)} + \int_0^t \frac{d\hat{a}}{dt} dt = \hat{a}^{(0)} - i\kappa \sum_j \hat{\pi}_j^{(0)} \frac{e^{-i(E_j/\hbar - \omega)t} - 1}{E_j/\hbar - \omega} \\ \hat{\pi}_j^{(1)} = \hat{\pi}_j^{(0)} + \int_0^t \frac{d\hat{\pi}_j}{dt} dt = \hat{\pi}_j^{(0)} + i\kappa (\hat{\pi}_j^{(0)\dagger} \hat{\pi}_j^{(0)} - \hat{\pi}_j^{(0)} \hat{\pi}_j^{(0)\dagger}) \hat{a}^{(0)} \frac{e^{i(E_j/\hbar - \omega)t} - 1}{E_j/\hbar - \omega} \end{array} \right.$$



$$\frac{d\hat{a}^{(2)}}{dt} = -\kappa \sum_j \hat{\pi}_j^{(1)} e^{-i(E_j/\hbar - \omega)t} \quad \left(\hat{a}^{(2)}, \hat{\pi}_j^{(2)} : \text{second-order solutions} \right)$$

$$\hat{a}^{(2)} = \hat{a}^{(0)} + \int_0^t \frac{d\hat{a}}{dt} dt = \hat{a}^{(0)} - \kappa \int_0^t \left\{ \sum_j \hat{\pi}_j^{(1)} e^{-i(E_j/\hbar - \omega)t} \right\} dt$$

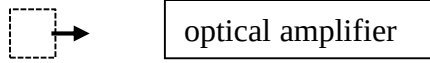
$$\begin{aligned}
&= \hat{a}^{(0)} - i\kappa \sum_j \hat{\pi}_j^{(0)} \frac{e^{-i(E_j/\hbar - \omega)t} - 1}{E_j/\hbar - \omega} \\
&\quad + i\hat{a}^{(0)}\kappa^2 \sum_j \left(\hat{\pi}_j^{(0)\dagger} \hat{\pi}_j^{(0)} - \hat{\pi}_j^{(0)} \hat{\pi}_j^{(0)\dagger} \right) \frac{1}{E_j/\hbar - \omega} \left\{ \frac{e^{i(E_j/\hbar - \omega)t} - 1}{-i(E_j/\hbar - \omega)} - t \right\} \\
&= \hat{a}^{(0)} [1 + \kappa^2 \sum_j \left(\hat{\pi}_j^{(0)\dagger} \hat{\pi}_j^{(0)} - \hat{\pi}_j^{(0)} \hat{\pi}_j^{(0)\dagger} \right) \left\{ \frac{1 - \cos[(E_j/\hbar - \omega)t]}{(E_j/\hbar - \omega)^2} - i \frac{t}{E_j/\hbar - \omega} \right\}] \\
&\quad - i\kappa \sum_j \hat{\pi}_j^{(0)} \frac{e^{-i(E_j/\hbar - \omega)t} - 1}{E_j/\hbar - \omega} \\
&= \hat{a}^{(0)} (1 + \hat{\Pi}) + \hat{P} \left(\begin{array}{l} \hat{\Pi} \equiv \kappa^2 \sum_j \left(\hat{\pi}_j^{(0)\dagger} \hat{\pi}_j^{(0)} - \hat{\pi}_j^{(0)} \hat{\pi}_j^{(0)\dagger} \right) \left\{ \frac{1 - \cos[(E_j/\hbar - \omega)t]}{(E_j/\hbar - \omega)^2} - i \frac{t}{E_j/\hbar - \omega} \right\} \\ \hat{P} \equiv -i\kappa \sum_j \hat{\pi}_j^{(0)} \frac{e^{-i(E_j/\hbar - \omega)t} - 1}{E_j/\hbar - \omega} \end{array} \right)
\end{aligned}$$

We regard this second-order solution as the time evolution during a short time τ .

$$\hat{a}(t_0 + \tau) = \hat{a}(t_0)(1 + \hat{\Pi}) + \hat{P}$$

In the following, we derive temporal differential equations for the mean values of physical quantities (i.e., mean amplitude, mean photon number, their variance), using the above equation.

Then, we calculate the time evolution of the physical quantities in a virtual space moving with the speed of light from the input to output ports of an optical amplifier.



For calculating the mean value of a physical quantity in the Heisenberg picture, we need the initial state. The averaged value of a physical quantity is given by the inner produce with an initial state $|\Psi(t_0)\rangle$.

Here, we assume that the atom state is uniform along the amplifier length.

Subsequently, an initial state is expressed as

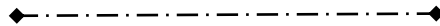
$$|\Psi(t_0)\rangle = \underbrace{|\Psi_R(t_0)\rangle}_{\text{light}} \otimes \underbrace{|\Psi_A(t_0)\rangle}_{\text{atoms}} = \underbrace{|\Psi_R(t_0)\rangle \otimes \{ \otimes_j (c_1|\phi_1\rangle + c_2|\phi_2\rangle) \}}_{|\Psi(t_0)\rangle}$$

Mean amplitude

The short-time evolution of the mean amplitude of light $\bar{a}$ is

$$\bar{a}(t_0 + \tau) = \langle \Psi(t_0) | \hat{a}(t_0 + \tau) | \Psi(t_0) \rangle = \langle \Psi(t_0) | \{ \hat{a}(t_0)(1 + \hat{\Pi}) + \hat{P} \} | \Psi(t_0) \rangle$$

$$\begin{aligned}
&= \langle \Psi_R(t_0) | \hat{a}(t_0) | \Psi_R(t_0) \rangle \langle \Psi_A | (1 + \hat{\Pi}) | \Psi_A \rangle + \langle \Psi_A | \hat{P} | \Psi_A \rangle \\
&= \bar{a}(t_0) \{ 1 + \langle \Psi_A | \hat{\Pi} | \Psi_A \rangle \} + \langle \Psi_A | \hat{P} | \Psi_A \rangle
\end{aligned}$$



$$\begin{aligned}
\langle \Psi_A | \hat{\Pi} | \Psi_A \rangle &= \langle \Psi_A | [\kappa^2 \sum_j (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) \left\{ \frac{1 - \cos[(E_j/\hbar - \omega)\tau]}{(E_j/\hbar - \omega)^2} - i \frac{\tau}{E_j/\hbar - \omega} \right\}] | \Psi_A \rangle \\
&= \kappa^2 \sum_j \langle \Psi_A | (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) | \Psi_A \rangle \left\{ \frac{1 - \cos[(E_j/\hbar - \omega)\tau]}{(E_j/\hbar - \omega)^2} - i \frac{\tau}{E_j/\hbar - \omega} \right\}
\end{aligned}$$

$$\begin{aligned}
\langle \phi_1 | \hat{\pi}_j \hat{\pi}_j^\dagger | \phi_1 \rangle_k &= \delta_{jk} \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle \langle \phi_1 | \phi_1 \rangle = \delta_{jk} \\
\langle \phi_2 | \hat{\pi}_j \hat{\pi}_j^\dagger | \phi_2 \rangle_k &= \delta_{jk} \langle \phi_2 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle \langle \phi_1 | \phi_2 \rangle = 0 \\
\langle \phi_1 | \hat{\pi}_j \hat{\pi}_j^\dagger | \phi_2 \rangle_k &= \delta_{jk} \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle \langle \phi_1 | \phi_2 \rangle = 0 \\
\langle \phi_2 | \hat{\pi}_j \hat{\pi}_j^\dagger | \phi_1 \rangle_k &= \delta_{jk} \langle \phi_2 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle \langle \phi_1 | \phi_1 \rangle = 0
\end{aligned}$$



$$\begin{aligned}
\langle \Psi_a | \hat{\pi}_j^\dagger \hat{\pi}_j | \Psi_a \rangle &= |c_2^{(j)}|^2 \\
\langle \Psi_a | \hat{\pi}_j \hat{\pi}_j^\dagger | \Psi_a \rangle &= |c_1^{(j)}|^2
\end{aligned}$$

$$= \kappa^2 \sum_j \left(|c_2^{(j)}|^2 - |c_1^{(j)}|^2 \right) \left\{ \frac{1 - \cos[(E_j/\hbar - \omega)\tau]}{(E_j/\hbar - \omega)^2} - i \frac{\tau}{E_j/\hbar - \omega} \right\} = \kappa^2 (N_2 - N_1) \pi \tau$$

$$(|c_2|^2 - |c_1|^2) \frac{1 - \cos[(E_j/\hbar - \omega)\tau]}{(E_j/\hbar - \omega)^2}$$

$$= \sum_j (|c_2|^2 - |c_1|^2) \frac{2 \sin^2[(E_j/\hbar - \omega)\tau/2]}{(E_j/\hbar - \omega)^2}$$

$$\left(\begin{array}{l} \text{Energy level } E_j \text{ is densely distributed} \\ \text{so that we regard it as a continuous variable } E. \end{array} \right)$$

$$= 2 \int_0^\infty (|c_2|^2 - |c_1|^2) \frac{\sin^2[(E/\hbar - \omega)\tau/2]}{(E/\hbar - \omega)^2} \rho(E) dE \quad \left[\rho(E): \text{distribution density of } E \right]$$

$$= 2 \int_{-\infty}^\infty (|c_2^{(j)}|^2 - |c_1^{(j)}|^2) \frac{\sin^2[\Omega \tau/2]}{\Omega^2} \rho(\Omega) d\Omega \quad \left[\Omega \equiv E/\hbar - \omega \right]$$

$$\left(\begin{array}{l} \rho(\Omega) \text{ is regarded as a constant, compared with the variation of } \sin^2[\Omega t/2]/\Omega^2. \\ (|c_2^{(j)}|^2 - |c_1^{(j)}|^2) \text{ is also regarded identical for each } j. \end{array} \right)$$

$$= 2(|c_2|^2 - |c_1|^2) \rho \int_{-\infty}^\infty \frac{\sin^2[\Omega \tau/2]}{\Omega^2} d\Omega = 2(N_2 - N_1) \frac{\pi \tau}{2} = (N_2 - N_1) \pi \tau$$

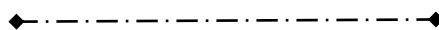
$$\left(\begin{array}{l} N_1: \text{The number of atoms at the lower state around } \Omega = 0 \\ N_2: \text{The number of atoms at the higher state around } \Omega = 0 \end{array} \right) \quad \text{i.e., } E \approx \hbar \omega$$

$$\sum_j (|c_2|^2 - |c_1|^2) \frac{\tau}{E_j/\hbar - \omega} = \int_{-\infty}^\infty (|c_2|^2 - |c_1|^2) \frac{\tau}{\Omega} \rho(\Omega) d\Omega = 0$$

The imaginary part is an odd function around $\Omega = 0 \rightarrow$ zero as a result of integration.

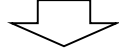
$$\langle \Psi_a | \hat{P} | \Psi_a \rangle = -i\kappa \langle \Psi_a | \sum_j \hat{\pi}_j^{(0)} \frac{e^{-i(E_j/\hbar - \omega)\tau} - 1}{E_j/\hbar - \omega} | \Psi_a \rangle = 0$$

$$\left(\begin{array}{l} \langle \phi_1 | \hat{\pi} | \phi_1 \rangle = \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_1 \rangle = 0 \\ \langle \Psi_a(0) | \sum_j \hat{\pi}_j^{(0)} | \Psi_a(0) \rangle = 0 \end{array} \right)$$



$$\bar{a}(t_0 + \tau) = \bar{a}(t_0) + \bar{a}(t_0) \frac{g}{2} (N_2 - N_1) \tau \quad \left[g \equiv 2\kappa^2\pi : \text{gain coefficient} \right]$$

$$\text{On the other hand, } \bar{a}(t_0 + \tau) = \bar{a}(t_0) + \frac{d\bar{a}}{dt} \tau \quad \text{for } \tau \ll 1$$

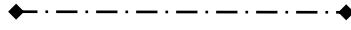


$$\frac{d\bar{a}}{dt} = \frac{g}{2} (N_2 - N_1) \bar{a} \longrightarrow \boxed{\bar{a}(t) = \bar{a}(0) e^{(g/2)(N_2 - N_1)t}}$$

Mean photon number

The short-time evolution of the mean photon number $\bar{n}$ is

$$\begin{aligned} \bar{n}(t_0 + \tau) &= \langle \Psi(t_0) | \hat{n}(t_0 + \tau) | \Psi(t_0) \rangle = \langle \Psi(t_0) | \hat{a}^\dagger(t_0 + \tau) \hat{a}(t_0 + \tau) | \Psi(t_0) \rangle \\ &= \langle \Psi(t_0) | \{ \hat{a}^\dagger(t_0) (1 + \hat{\Pi}^\dagger) + \hat{P}^\dagger \} \{ \hat{a}(t_0) (1 + \hat{\Pi}) + \hat{P} \} | \Psi(t_0) \rangle \\ &= \langle \Psi(t_0) | \{ \hat{a}^\dagger(t_0) \hat{a}(t_0) (1 + \hat{\Pi}^\dagger) (1 + \hat{\Pi}) + \hat{a}^\dagger(t_0) (1 + \hat{\Pi}^\dagger) \hat{P} + \hat{a}(t_0) \hat{P}^\dagger (1 + \hat{\Pi}) + \hat{P}^\dagger \hat{P} \} | \Psi(t_0) \rangle \\ &= \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi}^\dagger) (1 + \hat{\Pi}) | \Psi_a \rangle \\ &\quad + \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi}^\dagger) \hat{P} | \Psi_a \rangle \\ &\quad + \langle \Psi_r(t_0) | \hat{a}(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | \hat{P}^\dagger (1 + \hat{\Pi}) | \Psi_a \rangle \\ &\quad + \langle \Psi_a | \hat{P}^\dagger \hat{P} | \Psi_a \rangle \\ &= \bar{n}(t_0) \langle \Psi_a | (1 + \hat{\Pi}^\dagger) (1 + \hat{\Pi}) | \Psi_a \rangle + \bar{a}^*(t_0) \langle \Psi_a | (1 + \hat{\Pi}^\dagger) \hat{P} | \Psi_a \rangle \\ &\quad + \bar{a}(t_0) \langle \Psi_a | \hat{P}^\dagger (1 + \hat{\Pi}) | \Psi_a \rangle + \langle \Psi_a | \hat{P}^\dagger \hat{P} | \Psi_a \rangle \end{aligned}$$



$$\langle \Psi_a | (1 + \hat{\Pi}^\dagger) (1 + \hat{\Pi}) | \Psi_a \rangle \approx \langle \Psi_a | (1 + \hat{\Pi}^\dagger + \hat{\Pi}) | \Psi_a \rangle \quad \left[\text{Higher-order terms are neglected.} \right]$$

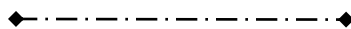
$$= \langle \Psi_a | \{ 1 + 2\kappa^2 \sum_j (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) \frac{1 - \cos[(E_j/\hbar - \omega)\tau]}{(E_j/\hbar - \omega)^2} \} | \Psi_a \rangle = 1 + g(N_2 - N_1)\tau$$

$$\begin{aligned} \langle \Psi_a | (1 + \hat{\Pi}^\dagger) \hat{P} | \Psi_a \rangle &= 0 \\ \langle \Psi_a | \hat{P}^\dagger (1 + \hat{\Pi}) | \Psi_a \rangle &= 0 \end{aligned} \quad \left(\begin{aligned} (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger)^\dagger &= (\hat{\pi}_j^\dagger \hat{\pi}_j - \hat{\pi}_j \hat{\pi}_j^\dagger) \\ \hat{\Pi}^\dagger &= \hat{\Pi} \end{aligned} \right)$$

$$\langle \Psi_a | \hat{P}^\dagger \hat{P} | \Psi_a \rangle = \kappa^2 \langle \Psi_a | \{ \sum_j \hat{\pi}_j^{(0)\dagger} \frac{e^{i(E_j/\hbar - \omega)\tau} - 1}{E_j/\hbar - \omega} \} \{ \sum_j \hat{\pi}_j^{(0)} \frac{e^{-i(E_j/\hbar - \omega)\tau} - 1}{E_j/\hbar - \omega} \} | \Psi_a \rangle$$

$$= \kappa^2 \sum_j \langle \Psi_a | \hat{\pi}_j^{(0)\dagger} \hat{\pi}_j^{(0)} | \Psi_a \rangle \frac{|e^{i(E_j/\hbar - \omega)\tau} - 1|^2}{(E_j/\hbar - \omega)^2} = gN_2\tau$$

$$\left[|e^{i(E_j/\hbar - \omega)\tau} - 1|^2 = 2 - 2 \cos[(E_j/\hbar - \omega)\tau] = 2\{1 - \cos[(E_j/\hbar - \omega)\tau]\} \right]$$



$$\bar{n}(t_0 + \tau) = \bar{n}(t_0) + \bar{n}(t_0)g(N_2 - N_1)\tau + gN_2\tau$$

Each term can be interpreted as follows:

$\bar{n}(t_0)$: Initial photon number

$\bar{n}(t_0)gN_2\tau$: Photon-number increase proportional to n and N_2

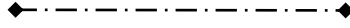
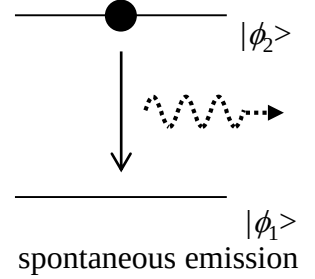
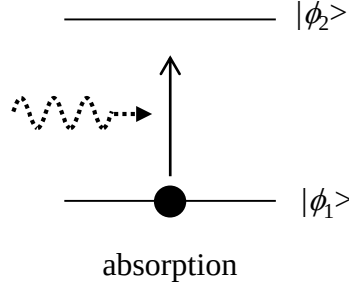
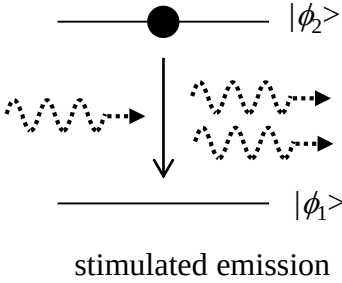
stimulated emission

$-\bar{n}(t_0)gN_1\tau$: Photon-number decrease proportional to n and N_1

absorption

$+gN_2\tau$: Photon-number increase proportional to N_2

spontaneous emission



On the other hand, $\bar{n}(t_0 + \tau) = \bar{n}(t_0) + \frac{d\bar{n}}{dt}\tau$

Thus, $\frac{d\bar{n}}{dt} = \boxed{g(N_2 - N_1)\bar{n}} + \boxed{gN_2}$ ← locally generated photons

↑
amplification
with gain coefficient $g(N_2 - N_1)$

The solution is:

$$\bar{n}(t) = \bar{n}(0)e^{g(N_2-N_1)t} + g \int_0^t N_2(t_0)e^{g(N_2-N_1)(t-t_0)} dt_0 = \boxed{G\bar{n}(0)} + \boxed{(G-1)n_{sp}}$$

↑ ↑
amplified signal spontaneous emission

where

$G \equiv e^{g(N_2-N_1)t}$: signal gain

$$g \int_0^t N_2(t_0)e^{g(N_2-N_1)(t-t_0)} dt_0 = gN_2 \frac{e^{g(N_2-N_1)t}-1}{g(N_2-N_1)} = (G-1)n_{sp}$$

$n_{sp} \equiv \frac{N_2}{N_2-N_1}$: population inversion parameter (noise factor)

Amplitude fluctuation

Fluctuation of the light amplitude can be evaluated by the variances of the real and imaginary parts of the annihilation operator (Chap. IV).

$$\begin{cases} \hat{a} = \hat{x}_1 + i\hat{x}_2 \\ \hat{a}^\dagger = \hat{x}_1 - i\hat{x}_2 \end{cases} \Rightarrow \begin{cases} \hat{x}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \rightarrow \sigma_{x1}^2 = \langle \hat{x}_1^2 \rangle - \langle \hat{x}_1 \rangle^2 & \text{(real part)} \\ \hat{x}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \rightarrow \sigma_{x2}^2 = \langle \hat{x}_2^2 \rangle - \langle \hat{x}_2 \rangle^2 & \text{(imaginary part)} \end{cases}$$

They are calculated as follows.

$$\begin{aligned}
\langle \hat{x}_1^2(t_0 + \tau) \rangle &= \frac{1}{4} \langle \Psi_0 | \{ \hat{a}(t_0 + \tau) + \hat{a}^\dagger(t_0 + \tau) \} \{ \hat{a}(t_0 + \tau) + \hat{a}^\dagger(t_0 + \tau) \} | \Psi_0 \rangle \\
&= \frac{1}{4} \{ \langle \Psi_0 | \hat{a}(t_0 + \tau) \hat{a}(t_0 + \tau) | \Psi_0 \rangle + \langle \Psi_0 | \hat{a}(t_0 + \tau) \hat{a}^\dagger(t_0 + \tau) | \Psi_0 \rangle \\
&\quad + \langle \Psi_0 | \hat{a}^\dagger(t_0 + \tau) \hat{a}(t_0 + \tau) | \Psi_0 \rangle + \langle \Psi_0 | \hat{a}^\dagger(t_0 + \tau) \hat{a}^\dagger(t_0 + \tau) | \Psi_0 \rangle \}
\end{aligned}$$

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$$\begin{aligned}
\langle \Psi_0 | \hat{a}(t_0 + \tau) \hat{a}(t_0 + \tau) | \Psi_0 \rangle &= \langle \Psi_0 | \{ \hat{a}(t_0)(1 + \hat{\Pi}) + \hat{P} \} \{ \hat{a}(t_0)(1 + \hat{\Pi}) + \hat{P} \} | \Psi_0 \rangle \\
&= \langle \Psi_r(t_0) | \hat{a}(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi})(1 + \hat{\Pi}) | \Psi_a \rangle \\
&\quad + \underbrace{\langle \Psi_r(t_0) | \hat{a}(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi}) \hat{P} + \hat{P} (1 + \hat{\Pi}) | \Psi_a \rangle}_0 + \underbrace{\langle \Psi_a | \hat{P} \hat{P} | \Psi_a \rangle}_0 \\
&\approx \langle \Psi_r(t_0) | \hat{a}(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\} \\
\langle \Psi_0 | \hat{a}(t_0 + \tau) \hat{a}^\dagger(t_0 + \tau) | \Psi_0 \rangle &= \langle \Psi_0 | \hat{a}(t_0) \{ (1 + \hat{\Pi}) + \hat{P} \} \hat{a}^\dagger(t_0) \{ (1 + \hat{\Pi}^\dagger) + \hat{P}^\dagger \} | \Psi_0 \rangle \\
&= \langle \Psi_r(t_0) | \hat{a}(t_0) \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi})(1 + \hat{\Pi}^\dagger) | \Psi_a \rangle \\
&\quad + \underbrace{\langle \Psi_r(t_0) | \hat{a}(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi}) \hat{P}^\dagger | \Psi_a \rangle}_0 \\
&\quad + \underbrace{\langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | \hat{P} (1 + \hat{\Pi}^\dagger) | \Psi_a \rangle}_0 + \langle \Psi_a | \hat{P} \hat{P}^\dagger | \Psi_a \rangle
\end{aligned}$$

$$\approx \langle \Psi_r(t_0) | \hat{a}(t_0) \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\} + gN_1\tau$$

$$\left(\begin{aligned}
\langle \Psi_a | \hat{P} \hat{P}^\dagger | \Psi_a \rangle &= \kappa^2 \langle \Psi_a | \left\{ \sum_j \hat{n}_j \frac{e^{-i(E_j/\hbar - \omega)\tau} - 1}{E_j/\hbar - \omega} \right\} \left\{ \sum_j \hat{n}_j^\dagger \frac{e^{i(E_j/\hbar - \omega)\tau} - 1}{E_j/\hbar - \omega} \right\} | \Psi_a \rangle \\
&= \kappa^2 \sum_j \langle \Psi_a | \hat{n}_j \hat{n}_j^\dagger | \Psi_a \rangle \frac{|e^{i(E_j/\hbar - \omega)\tau} - 1|^2}{(E_j/\hbar - \omega)^2} = gN_1\tau
\end{aligned} \right)$$

$$\begin{aligned}
\langle \Psi_0 | \hat{a}^\dagger(t_0 + \tau) \hat{a}(t_0 + \tau) | \Psi_0 \rangle &= \langle \Psi_0 | \hat{a}^\dagger(t_0) \{ (1 + \hat{\Pi}^\dagger) + \hat{P}^\dagger \} \hat{a}(t_0) \{ (1 + \hat{\Pi}) + \hat{P} \} | \Psi_0 \rangle \\
&= \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi}^\dagger)(1 + \hat{\Pi}) | \Psi_a \rangle + \langle \Psi_a | \hat{P}^\dagger \hat{P} | \Psi_a \rangle \\
&\approx \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\} + gN_2\tau \\
\langle \Psi_0 | \hat{a}^\dagger(t_0 + \tau) \hat{a}^\dagger(t_0 + \tau) | \Psi_0 \rangle &= \langle \Psi_0 | \hat{a}^\dagger(t_0) \{ (1 + \hat{\Pi}^\dagger) + \hat{P}^\dagger \} \hat{a}^\dagger(t_0) \{ (1 + \hat{\Pi}^\dagger) + \hat{P}^\dagger \} | \Psi_0 \rangle \\
&= \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \langle \Psi_a | (1 + \hat{\Pi}^\dagger)(1 + \hat{\Pi}^\dagger) | \Psi_a \rangle + \langle \Psi_a | \hat{P}^\dagger \hat{P}^\dagger | \Psi_a \rangle \\
&\approx \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\}
\end{aligned}$$

◆.....◆

$$\begin{aligned}
\langle \hat{x}_1^2(t_0 + \tau) \rangle &= \frac{1}{4} [\langle \Psi_r(t_0) | \hat{a}(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\} \\
&\quad + \langle \Psi_r(t_0) | \hat{a}(t_0) \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\} + gN_1\tau \\
&\quad + \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\} + gN_2\tau \\
&\quad + \langle \Psi_r(t_0) | \hat{a}^\dagger(t_0) \hat{a}^\dagger(t_0) | \Psi_r(t_0) \rangle \{1 + g(N_2 - N_1)\tau\}]
\end{aligned}$$

$$=< \Psi_r(t_0) | \hat{x}_1^2(t_0) | \Psi_r(t_0) > \{1 + g(N_2 - N_1)\tau\} + \frac{1}{4}g(N_1 + N_2)\tau$$

On the other hand,

$$\begin{aligned} < \hat{x}_1(t_0 + \tau) >^2 &= \frac{1}{4} \{ < \Psi_0 | \hat{a}(t_0 + \tau) + \hat{a}^\dagger(t_0 + \tau) | \Psi_0 > \}^2 \\ &= \frac{1}{4} \{ < \Psi_0 | \hat{a}(t_0 + \tau) | \Psi_0 > + < \Psi_0 | \hat{a}^\dagger(t_0 + \tau) | \Psi_0 > \}^2 \\ &= \frac{1}{4} [< \Psi_r(t_0) | \hat{a}(t_0) | \Psi_r(t_0) > \{1 + \frac{g}{2}(N_2 - N_1)\tau\} \\ &\quad + < \Psi_r(t_0) | \hat{a}^\dagger(t_0) | \Psi_r(t_0) > \{1 + \frac{g}{2}(N_2 - N_1)\tau\}]^2 \\ &= \frac{1}{4} \{ < \Psi_r(t_0) | \hat{a}(t_0) | \Psi_r(t_0) > + < \Psi_r(t_0) | \hat{a}^\dagger(t_0) | \Psi_r(t_0) > \}^2 \{1 + \frac{g}{2}(N_2 - N_1)\tau\}^2 \\ &\approx \frac{1}{4} \{ < \Psi_r(t_0) | \{ \hat{a}(t_0) + \hat{a}^\dagger(t_0) \} | \Psi_r(t_0) > \}^2 \{1 + g(N_2 - N_1)\tau\} \\ &= \{ < \Psi_r(t_0) | \hat{x}_1(t_0) | \Psi_r(t_0) > \}^2 \{1 + g(N_2 - N_1)\tau\} \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{x1}^2(t_0 + \tau) &= < \hat{x}_1^2(t_0 + \tau) > - < \hat{x}_1(t_0 + \tau) >^2 \\ &= < \Psi_r(t_0) | \hat{x}_1^2(t_0) | \Psi_r(t_0) > \{1 + g(N_2 - N_1)\tau\} + \frac{1}{4}g(N_1 + N_2)\tau \\ &\quad - \{ < \Psi_r(t_0) | \hat{x}_1(t_0) | \Psi_r(t_0) > \}^2 \{1 + g(N_2 - N_1)\tau\} \\ &= [< \Psi_r(t_0) | \hat{x}_1^2(t_0) | \Psi_r(t_0) > - \{ < \Psi_r(t_0) | \hat{x}_1(t_0) | \Psi_r(t_0) > \}^2] \{1 + g(N_2 - N_1)\tau\} \\ &\quad + \frac{1}{4}g(N_1 + N_2)\tau \\ &= \sigma_{x1}^2(t_0) \{1 + g(N_2 - N_1)\tau\} + \frac{1}{4}g(N_1 + N_2)\tau \end{aligned}$$

From this equation, the following differential equation is obtained.

$$\begin{aligned} \frac{d\sigma_{x1}^2}{dt} &= g(N_2 - N_1)\sigma_{x1}^2 + \frac{1}{4}g(N_1 + N_2) \\ &\quad \Downarrow \\ \sigma_{x1}^2(t) &= \sigma_{x1}^2(0)e^{g(N_2 - N_1)t} + \frac{1}{4}g(N_1 + N_2) \int_0^t e^{g(N_2 - N_1)(t-t_0)} dt_0 \\ &= \sigma_{x1}^2(0)e^{g(N_2 - N_1)t} + \frac{1}{4}g(N_1 + N_2) \frac{e^{g(N_2 - N_1)t} - 1}{g(N_1 - N_2)} \\ &= G\sigma_{x1}^2(0) + \frac{1}{4} \frac{N_1 + N_2}{N_1 - N_2} (G - 1) \\ &= \boxed{G\sigma_{x1}^2(0)} + \boxed{\frac{1}{4}(2n_{sp} - 1)(G - 1)} \quad \left(\frac{N_1 + N_2}{N_1 - N_2} = \frac{2N_1}{N_1 - N_2} - 1 = 2n_{sp} - 1 \right) \\ \text{amplified noise} &\quad \text{additional noise} \end{aligned}$$

In summary,

amplitude

$$\bar{a}_{\text{out}} = \sqrt{G} \bar{a}_{\text{in}}$$

photon number

$$\bar{n}_{\text{out}} = G \bar{n}_{\text{in}} + (G - 1) n_{\text{sp}}$$

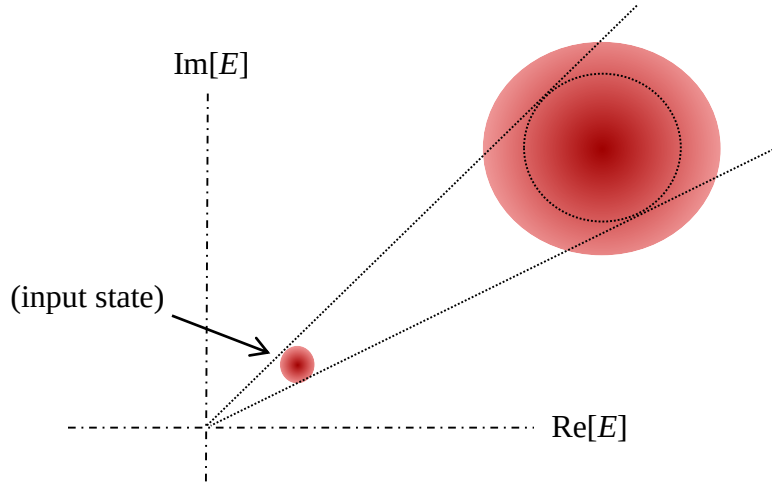
(amplified photon) (spontaneous emission)

amplitude fluctuation

$$\sigma_{x1}^2(\text{out}) = G \sigma_{x1}^2(\text{in}) + \frac{G-1}{4} (2n_{\text{sp}} - 1)$$

(amplified noise) (additional noise)

The above amplification process can be illustrated in the complex amplitude space (constellation) as



Here, the input light is supposed to be a pure coherent state.

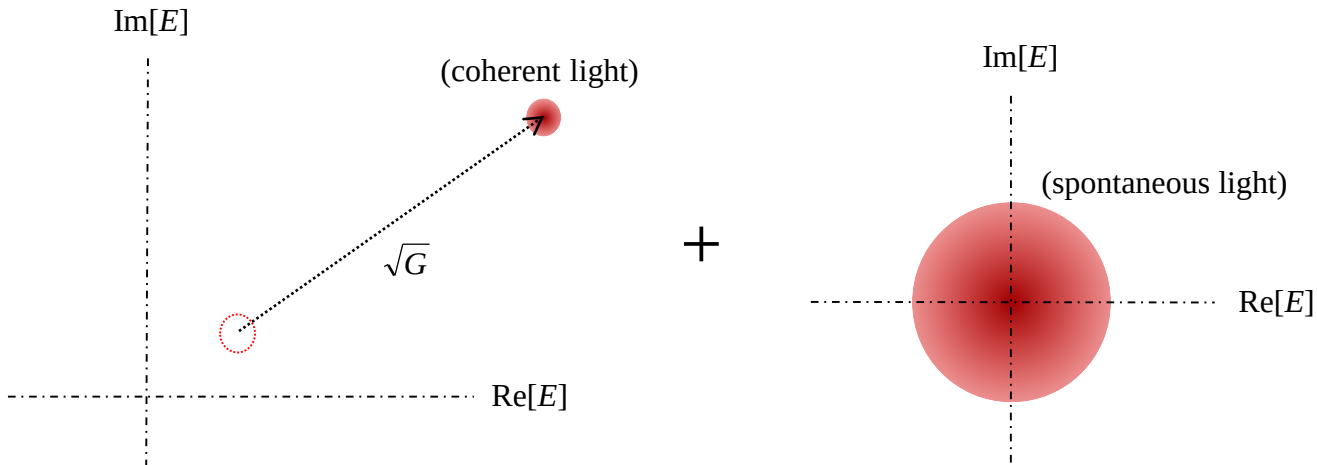
In this case, the amplitude variances of the input light are: $\sigma_{x1}^2(\text{in}) = \sigma_{x2}^2(\text{in}) = \frac{1}{4}$

Then, the amplitude variances of the output light are:

$$\sigma_{x1}^2(\text{out}) = \sigma_{x2}^2(\text{out}) = \frac{1}{4} G + \frac{G-1}{4} (2n_{\text{sp}} - 1) = \underbrace{\frac{1}{4}}_{\text{(var. of a coherent state)}} + \underbrace{\frac{1}{2} (G-1) n_{\text{sp}}}_{\text{(additional variance)}}$$

This expressions can be interpreted as:

“the amplified light is composed of a coherent light and spontaneously emitted light”.



A coherent state corresponds to a noise-free light in the classical world.

Thus, the above consideration suggests that the transfer function of an optical amplifier can be classically expressed as:

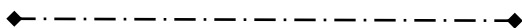
$$E_{\text{out}} = \boxed{\sqrt{G}E_{\text{in}}} + \boxed{E_{\text{sp}}}$$

↑ amplified signal light ↑ spontaneously emitted light
(or amplified spontaneous emission: ASE)

where the mean value and the variance of ASE light are given by

$$\begin{aligned} \langle E_{\text{sp}} \rangle &= 0 \\ \langle \{ \text{Re}[E_{\text{sp}}] \}^2 \rangle &= \langle \{ \text{Im}[E_{\text{sp}}] \}^2 \rangle = \frac{1}{2} (G-1) n_{\text{sp}} \times (hf \times \Delta f_{\text{sp}}) \end{aligned} \quad \left(\begin{array}{l} hf : \text{one-photon energy} \\ \Delta f_{\text{sp}} : \text{frequency bandwidth of ASE} \end{array} \right)$$

$\left(\begin{array}{l} hf \text{ is multiplied because the energy unit in the above discussion is one photon.} \\ \Delta f_{\text{sp}} \text{ is multiplied because the above discussion is for one resonance mode.} \end{array} \right)$



Amplitude fluctuation is evaluated in the above.

Photon number fluctuation (i.e., intensity fluctuation) can be also evaluated with similar calculations.

However, the derivation is rather complicated, and we just show results below.

$$\begin{aligned} \sigma_n^2(\text{out}) &= \langle \hat{n}(\text{out})^2 \rangle - \langle \hat{n}(\text{out}) \rangle^2 = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}(\text{out}) \rangle - \langle \hat{a}^\dagger \hat{a}(\text{out}) \rangle^2 \\ &= \boxed{G n_{\text{in}}} + \boxed{(G-1) n_{\text{sp}}} + \boxed{2G(G-1) n_{\text{in}} n_{\text{sp}}} + \boxed{(G-1)^2 n_{\text{sp}}^2} + \boxed{G^2 \{ \langle \hat{n}^2(\text{in}) \rangle - n_{\text{in}}^2 - n_{\text{in}} \}} \end{aligned}$$

↑ signal shot noise ↑ spontaneous shot noise ↑ signal-spontaneous beat noise ↑ spontaneous-spontaneous beat noise ↑ amplified excess noise

Noise Figure (NF)

The noise performance of an amplifier is usually indicated by “noise figure (NF)”.

We describe NF of an optical amplifier in terms of quantum mechanics in this section.

The noise figure is generally defined as:

$$NF \equiv \frac{(SNR)_{\text{in}}}{(SNR)_{\text{out}}} = \frac{(S/N)_{\text{in}}}{(S/N)_{\text{out}}} \quad (SNR)_{\text{in}} \Rightarrow \triangle \Rightarrow (SNR)_{\text{out}}$$

where SNR: Signal-to-Noise Ratio, S: Signal power, N: Noise power

Historically, SNR is evaluated in terms of the photo-current generating in direct detection, which is equivalent to SNR in the photon number.



In our situation,

$$\begin{cases} S = (\text{mean photon number})^2 = \langle \hat{n} \rangle^2 \\ N = \text{the variance of the photon number} = \sigma_n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \end{cases}$$

From the previous results,

$$\begin{cases} S_{\text{out}} = \langle \hat{n}(\text{out}) \rangle^2 = \{Gn_{\text{in}} + (G-1)n_{\text{sp}}\}^2 \\ N_{\text{out}} = \sigma_n^2 = Gn_{\text{in}} + (G-1)n_{\text{sp}} + 2G(G-1)n_{\text{in}}n_{\text{sp}} + (G-1)^2n_{\text{sp}}^2 + G^2\{\langle \hat{n}^2(\text{in}) \rangle - n_{\text{in}}^2\} \end{cases}$$

Usually, $\{n_0 \gg n_{\text{sp}}, G \gg 1\}$ and then the above expressions are approximated as:

$$\begin{cases} S_{\text{out}} \approx G^2n_{\text{in}}^2 \\ N_{\text{out}} \approx 2G(G-1)n_{\text{in}}n_{\text{sp}} \approx 2G^2n_{\text{in}}n_{\text{sp}} \quad (\text{i.e., signal-spontaneous beat noise is dominant}) \end{cases}$$

Then,

$$(SNR)_{\text{out}} = \frac{G^2n_0^2}{2G^2n_0n_{\text{sp}}} = \frac{n_0}{2n_{\text{sp}}}$$

On the other hand, $(SNR)_{\text{in}}$ cannot be specifically expressed in general, because it depends on the input light state.

In order to indicate the intrinsic noise property of amplifiers, the input light is usually assumed to be ideally clean light, i.e., coherent state.

$$\begin{aligned} \begin{cases} S_{\text{in}} = n_{\text{in}}^2 \\ N_{\text{in}} = \langle \alpha | \hat{n}^2 | \alpha \rangle - \langle \alpha | \hat{n} | \alpha \rangle^2 = \langle \alpha | \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle - \langle \alpha | \hat{n} | \alpha \rangle^2 \\ \quad = \langle \alpha | \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) \hat{a} | \alpha \rangle - \langle \alpha | \hat{n} | \alpha \rangle^2 \\ \quad = \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \alpha \rangle + \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle - n_{\text{in}}^2 \\ \quad = |\alpha|^4 + \langle \alpha | \hat{n} | \alpha \rangle - n_{\text{in}}^2 \\ \quad = n_{\text{in}}^2 + n_{\text{in}} - n_{\text{in}}^2 = n_{\text{in}} \end{cases} \end{aligned}$$

Then,.

$$(SNR)_{\text{in}} = \frac{n_{\text{in}}^2}{n_{\text{in}}} = n_{\text{in}}$$

From the above equations,

$$NF = \frac{(SNR)_{\text{in}}}{(SNR)_{\text{out}}} = \frac{n_{\text{in}}}{n_{\text{in}} / 2n_{\text{sp}}} = 2n_{\text{sp}}$$

Usually, the dB-representation is used for the noise figure, $(NF)_{\text{dB}} = 10 \log_{10}(NF) = 10 \log_{10}(2n_{\text{sp}})$

By the way, n_{sp} in the above expression is:

$$n_{\text{sp}} = \frac{N_2}{N_2 - N_1} = \frac{1}{1 - N_1 / N_2} \geq 1 \quad \left(\begin{array}{l} N_2: \text{the number of atoms at the upper energy state} \\ N_1: \text{the number of atoms at the lower energy state} \end{array} \right)$$

Thus,

$$NF \geq 2 \quad (3 \text{ dB})$$

$NF = 3\text{dB}$ is the quantum-limited noise figure of amplifiers.

Any amplifier based on light-atom interaction cannot show better noise performance than $NF = 3\text{dB}$.