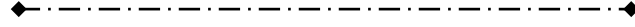


An ideal monochromatic electromagnetic wave (or laser light, approximately) is “coherent state” in terms of quantum mechanics.

Coherent states are inherently accompanied with fluctuation (or quantum noise), which determines the ultimate performance of optical systems, e.g., fiber communication and optical sensing.

This chapter describes coherent states.



An operator representing a monochromatic electromagnetic wave is

$$\hat{E}(z, t) = iA\{\hat{a}e^{i(kz - \omega t)} - \hat{a}^\dagger e^{-i(kz - \omega t)}\}$$

$$\left(A = \sqrt{\frac{\hbar \omega}{2\epsilon_0 V}} \right)$$

We first discuss about $\hat{a}$ (annihilation operator) in this expression, which has eigenstate/eigenvalue as

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad (\alpha \text{ is a complex number because } \hat{a} \text{ is not an Hermitian operator.})$$

The eigenstate can be expressed as a linear combination of photon-number states.

$$|\alpha\rangle = \sum_n c_n |n\rangle \quad \left(\text{Any state can be expressed by a linear combination of eigenstates of a physical quantity operator (i.e., Hermitian operator).} \right)$$

where $c_n = \langle n|\alpha\rangle$.

Coefficient c_n can be derived as follows.

Take the inner product of the eigenvalue/eigenstate equation with $|n\rangle$ as:

$$\begin{aligned} \hat{a}|\alpha\rangle &= \alpha|\alpha\rangle & \left(\langle \alpha|\hat{a}^\dagger = \langle \alpha|^* \right) \\ \langle n|\hat{a}|\alpha\rangle &= \alpha \langle n|\alpha\rangle \end{aligned}$$

$$\text{Recall here that } \langle n|\hat{a} = \sqrt{n+1}\langle n+1| \quad \left(\hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle \longrightarrow \langle n|\hat{a} = \sqrt{n+1}\langle n+1| \right)$$

Substitute this into the above equation:

$$\begin{aligned} \text{(left)} \quad & \langle n|\hat{a}|\alpha\rangle = \sqrt{n+1}\langle n+1|\alpha\rangle \\ \text{(right)} \quad & \alpha \langle n|\alpha\rangle \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{(left)} \quad & \langle n|\hat{a}|\alpha\rangle = \sqrt{n+1}\langle n+1|\alpha\rangle \\ \text{(right)} \quad & \alpha \langle n|\alpha\rangle \end{aligned}} \right\} \longrightarrow \begin{aligned} & \langle n+1|\alpha\rangle = \frac{\alpha}{\sqrt{n+1}} \langle n|\alpha\rangle \\ & \langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}} \langle n-1|\alpha\rangle \end{aligned}$$

$\langle n|\alpha\rangle$ is expressed by $\langle n-1|\alpha\rangle$ with coefficient $\alpha/\sqrt{n}$

The above expression can be consecutively rewritten as

$$\boxed{\langle n|\alpha\rangle} = \frac{\alpha}{\sqrt{n}} \langle n-1|\alpha\rangle = \frac{\alpha^2}{\sqrt{n(n-1)}} \langle n-2|\alpha\rangle = \dots = \frac{\alpha^n}{\sqrt{n!}} \langle 0|\alpha\rangle$$

$\parallel$
 c_n



$$|\alpha\rangle = \sum_n c_n |n\rangle = \langle 0|\alpha\rangle \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

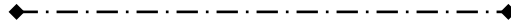
$\langle 0|\alpha\rangle$ in this expression is obtained from normalization as

$$\begin{aligned}
 \langle \alpha | \alpha \rangle &= \langle 0 | \alpha \rangle^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = \langle 0 | \alpha \rangle^2 \exp(|\alpha|^2) = 1 \\
 \langle 0 | \alpha \rangle^2 &= \exp(-|\alpha|^2) \\
 \langle 0 | \alpha \rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)
 \end{aligned}
 \quad \left(\begin{array}{l} \langle m | n \rangle = \delta_{mn} \\ e^x = \sum_n \frac{x^n}{n!} \end{array} \right)^2$$

Then,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

This state is called coherent state.



The eigenvalue of $\hat{a}$ is α .

Since $\hat{a}$ is not an Hermitian operator, α is a complex number and $\hat{a}$ is not observable. But, recalling that $\hat{a}$ is an operator originating from complex amplitude of lightwave, we can regard α as the complex amplitude of lightwave.

This claim may be not accurate from the rigid quantum mechanics theory.
Just an intuitive statement.

Monochromatic light is a coherent state

An operator representing a monochromatic electromagnetic field is

$$\hat{E}(z, t) = iA\{\hat{a}e^{i(kz - \omega t)} - \hat{a}^\dagger e^{-i(kz - \omega t)}\}$$

Let us consider the expected value of the above operator for a coherent state.

$$\begin{aligned}
 \langle \alpha | \hat{E}(z, t) | \alpha \rangle &= iA[\langle \alpha | \hat{a} | \alpha \rangle e^{i(kz - \omega t)} - \langle \alpha | \hat{a}^\dagger | \alpha \rangle e^{-i(kz - \omega t)}] \\
 &= iA\{\alpha e^{i(kz - \omega t)} - \alpha^* e^{-i(kz - \omega t)}\} \\
 &= iA\{|\alpha| e^{i(kz - \omega t + \theta)} - |\alpha| e^{-i(kz - \omega t + \theta)}\} \\
 &= 2|\alpha| A \sin(\omega t - kz - \theta)
 \end{aligned}$$

$$\left(\begin{array}{l} \hat{a} | \alpha \rangle = \alpha | \alpha \rangle \\ \downarrow \\ \langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha | \end{array} \right)$$

$(\alpha = |\alpha| e^{i\theta})$

This is identical to a classical expression of a monochromatic lightwave.



monochromatic light = coherent state

Light is usually measured in terms of intensity or power.

The intensity is proportional to the photon number.

Therefore, this section discusses the photon number of a coherent state.

A coherent state is expressed by a linear combination of photon-number states:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$|n\rangle$ is a n -photon state, whose probability is given by the absolute square of its coefficient.

Thus, when we measure the photon number of a coherent state, the probability of measuring n photons is given by

$$P(n) = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

The photon number is probabilistic, following

Poisson distribution

The average photon number is

$$\begin{aligned} \bar{n} &= \sum_{n=0}^{\infty} n P(n) = \sum_{n=0}^{\infty} n e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = \sum_{n=1}^{\infty} n e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = |\alpha|^2 \sum_{n=1}^{\infty} e^{-|\alpha|^2} \frac{|\alpha|^{2(n-1)}}{(n-1)!} \\ &= |\alpha|^2 \sum_{n=0}^{\infty} e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \\ &= |\alpha|^2 \left(\sum_n P(n) = 1 \right) \end{aligned}$$

Intuitively speaking, α corresponds to the complex amplitude of lightwave.

The above result corresponds to a classical picture that the light intensity is given by the absolute square of its complex amplitude.

On the other hand, the variance of the photon number is

$$\begin{aligned} \sigma_n^2 &= \langle n^2 \rangle - \langle n \rangle^2 = \sum_{n=0}^{\infty} n^2 P(n) - \bar{n}^2 = \sum_{n=0}^{\infty} n^2 e^{-\bar{n}} \frac{\bar{n}^n}{n!} - \bar{n}^2 = \bar{n} \sum_{n=1}^{\infty} n e^{-\bar{n}} \frac{\bar{n}^{(n-1)}}{(n-1)!} - \bar{n}^2 \\ &= \bar{n} \sum_{n=0}^{\infty} (n+1) e^{-\bar{n}} \frac{\bar{n}^n}{n!} - \bar{n}^2 = \bar{n} \sum_{n=0}^{\infty} n e^{-\bar{n}} \frac{\bar{n}^n}{n!} + \bar{n} \sum_{n=0}^{\infty} e^{-\bar{n}} \frac{\bar{n}^n}{n!} - \bar{n}^2 = \bar{n} \bar{n} + \bar{n} - \bar{n}^2 = \bar{n} \end{aligned}$$

The photon number of a coherent state is not deterministic, but stochastically distributes.

||
The photon number fluctuates.

||
The intensity of light fluctuates. → intensity noise

This fluctuation originates from the fact that a coherent state is a superposition state of photon-number states. In other words, it originates from the property of quantum states.

quantum noise

The previous section discusses the photon number of light (i.e., light intensity). This section considers the amplitude from the quantum mechanical viewpoint.

First, we decompose the annihilation/creation operators into the real and imaginary parts:

$$\begin{cases} \hat{a} = \hat{x}_1 + i\hat{x}_2 \\ \hat{a}^\dagger = \hat{x}_1 - i\hat{x}_2 \end{cases} \quad \begin{pmatrix} \hat{x}_1 : \text{real part operator} \\ \hat{x}_2 : \text{imaginary part operator} \end{pmatrix}$$

Using these operators, the operator representing an electromagnetic wave is rewritten as

$$\begin{aligned} \hat{E}(z, t) &= iA\{\hat{a}e^{i(kz-\omega t)} - \hat{a}^\dagger e^{-i(kz-\omega t)}\} \\ &= iA\{(\hat{x}_1 + i\hat{x}_2)e^{i(kz-\omega t)} - (\hat{x}_1 - i\hat{x}_2)e^{-i(kz-\omega t)}\} \\ &= iA[(\hat{x}_1 + i\hat{x}_2)\{\cos(kz - \omega t) + i\sin(kz - \omega t)\} - (\hat{x}_1 - i\hat{x}_2)\{\cos(kz - \omega t) - i\sin(kz - \omega t)\}] \\ &= -2A\{\hat{x}_1 \sin(kz - \omega t) + \hat{x}_2 \cos(kz - \omega t)\} \\ &= 2A\{\hat{x}_1 \sin(\omega t - kz) - \hat{x}_2 \cos(\omega t - kz)\} \end{aligned}$$



$$\begin{cases} \hat{x}_1 \text{ corresponds to the amplitude of the sine component} \\ \hat{x}_2 \text{ corresponds to the amplitude of the cosine component} \end{cases}$$

Orthogonal phase-amplitude operator

$\hat{x}_1$ and $\hat{x}_2$ are Hermitian operators.

$$\begin{cases} \hat{a} = \hat{x}_1 + i\hat{x}_2 \\ \hat{a}^\dagger = \hat{x}_1 - i\hat{x}_2 \end{cases} \Rightarrow \begin{cases} \hat{x}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \rightarrow \hat{x}_1^\dagger = \frac{1}{2}(\hat{a}^\dagger + \hat{a}) = \hat{x}_1 \\ \hat{x}_2 = \frac{i}{2}(\hat{a}^\dagger - \hat{a}) \rightarrow \hat{x}_2^\dagger = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger) = \hat{x}_2 \end{cases}$$

Thus, $\{\hat{x}_1, \hat{x}_2\}$ correspond to observable physical quantities.

How about the commutation relation ?

$$[\hat{x}_1, \hat{x}_2] = \hat{x}_1\hat{x}_2 - \hat{x}_2\hat{x}_1 = \frac{i}{4}\{(\hat{a} + \hat{a}^\dagger)(\hat{a}^\dagger - \hat{a}) - (\hat{a}^\dagger - \hat{a})(\hat{a} + \hat{a}^\dagger)\} = \frac{i}{2}(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}) = \frac{i}{2}$$

$\{\hat{x}_1, \hat{x}_2\}$ are in an uncertainty relationship.

$$\left[\hat{a}, \hat{a}^\dagger \right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

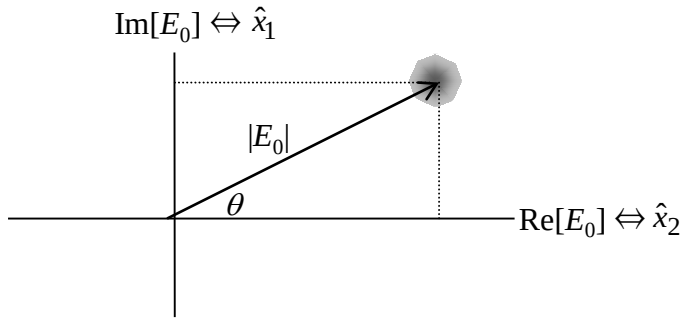
About sine & cosine components

Complex expression of light field: $\tilde{E} = E_0 \exp[i(kz - \omega t)]$ (E_0 : complex amplitude)

$$\begin{aligned} \text{Actual physical quantity: } E &= \text{Re}[\tilde{E}] = |E_0| \cos[kz - \omega t + \theta] \quad (E_0 = |E_0|e^{i\theta}) \\ &= |E_0| \{\cos(kz - \omega t) \cos(\theta) - \sin(kz - \omega t) \sin(\theta)\} \\ &= |E_0| \sin(\theta) \sin(\omega t - kz) - |E_0| \cos(\theta) \cos(\omega t - kz) \end{aligned}$$

$$= \boxed{\text{Im}[E_0]} \sin(\omega t - kz) - \boxed{\text{Re}[E_0]} \cos(\omega t - kz)$$

$$\hat{x}_1 \qquad \hat{x}_2$$



$[\hat{x}_1, \hat{x}_2] = \frac{i}{2} \neq 0$ suggests that the real and imaginary parts cannot be precisely measured simultaneously.



Amplitude fluctuation

Amplitude fluctuation of coherent state

The amplitude operators are in an uncertainty relationship, as described above.
How uncertain they are ?

Fluctuation is generally evaluated by the variance $\sigma^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$

In the present case,

$$\left\{ \begin{aligned} \langle \hat{x}_1 \rangle &= \langle \alpha | \hat{x}_1 | \alpha \rangle = \frac{1}{2} \{ \langle \alpha | \boxed{\hat{a}} | \alpha \rangle + \langle \alpha | \boxed{\hat{a}^\dagger} | \alpha \rangle \} = \frac{\alpha + \alpha^*}{2} \\ \langle \hat{x}_1^2 \rangle &= \langle \alpha | \hat{x}_1^2 | \alpha \rangle = \frac{1}{4} \langle \alpha | \hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger | \alpha \rangle \\ &= \frac{1}{4} \langle \alpha | \hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}^\dagger | \alpha \rangle = \frac{1}{4} \{ \alpha^2 + 2\alpha^*\alpha + 1 + (\alpha^*)^2 \} \end{aligned} \right.$$



$$\sigma_{x1}^2 = \langle \hat{x}_1^2 \rangle - \langle \hat{x}_1 \rangle^2 = \frac{1}{4} \{ \alpha^2 + 2\alpha^*\alpha + 1 + (\alpha^*)^2 - (\alpha + \alpha^*)^2 \} = \frac{1}{4}$$

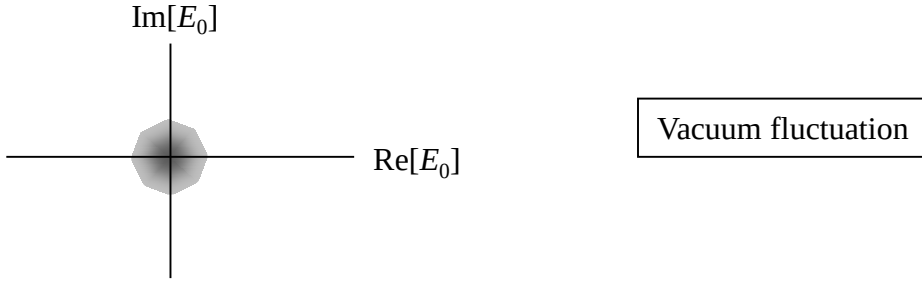
$$\left\{ \begin{aligned} \langle \hat{x}_2 \rangle &= \langle \alpha | \hat{x}_2 | \alpha \rangle = \frac{i}{2} (\alpha^* - \alpha) \\ \langle \hat{x}_2^2 \rangle &= \langle \alpha | \hat{x}_2^2 | \alpha \rangle = -\frac{1}{4} \langle \alpha | \hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a} | \alpha \rangle \\ &= -\frac{1}{4} \langle \alpha | \hat{a}^\dagger\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a} - 1 + \hat{a}\hat{a} | \alpha \rangle = -\frac{1}{4} \{ (\alpha^*)^2 - 2\alpha^*\alpha - 1 + \alpha^2 \} \end{aligned} \right.$$



$$\sigma_{x2}^2 = \langle \hat{x}_2^2 \rangle - \langle \hat{x}_2 \rangle^2 = -\frac{1}{4} \{ \alpha^2 - 2\alpha^*\alpha - 1 + (\alpha^*)^2 \} + (\alpha - \alpha^*)^2 = \frac{1}{4}$$

- ◆ The amplitude equally fluctuates in the two components.
- ◆ The variance is constant irrespective of the mean amplitude.

The amplitude variance of a coherent state is constant, irrespective of the mean value.
Thus, even the coherent state with zero mean amplitude has fluctuation.



The energy of the vacuum fluctuation corresponds to the minimum energy of electromagnetic wave introduced in Chap. 3.

$$\hat{H} | n \rangle = \hbar\omega\left(n + \frac{1}{2}\right) | n \rangle$$

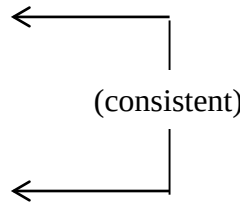
$\left(\begin{array}{l} n = \{0, 1, 2, 3, \dots\} \text{ represents the photon number.} \\ \hbar\omega \text{ is the one-photon energy.} \end{array} \right)$

$$E_0 = \frac{\hbar\omega}{2}$$

In the amplitude of a coherent state:

$$\sigma_{x1}^2 + \sigma_{x2}^2 = \frac{1}{2}$$

(in the unit of one-photon energy)



(Note: the variance = the noise power)

Relation to “Uncertainty principle” in Chap. 2.

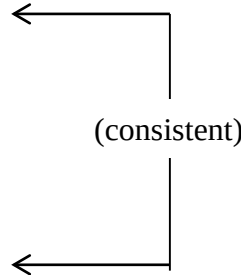
The degree of the uncertainty between two physical quantities is given by

$$\langle (\Delta\hat{A})^2 \rangle \langle (\Delta\hat{B})^2 \rangle \geq \frac{|\langle [\hat{A}, \hat{B}] \rangle|^2}{4}$$

In the light amplitude of a coherent state:

$$\left. \frac{|\langle [\hat{x}_1, \hat{x}_2] \rangle|^2}{4} = \frac{|i/2|^2}{4} = \frac{1}{16} \right\}$$

$$\{\sigma_{x1}^2 = \frac{1}{4}, \sigma_{x2}^2 = \frac{1}{4}\}$$



Quantum noise vs classical noise (1)

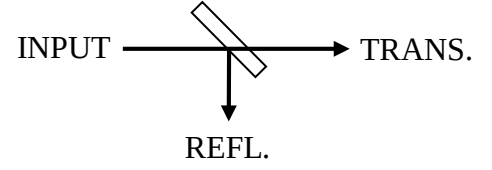
7

Actual lightwave fluctuates due to practical factors (e.g., pump fluctuation, reflection) other than the quantum noise. Such noises are generally called “classical noise”.

There is intrinsic differences between the quantum noise and the classical noise.

Lightwave is assumed to be split into two by a beam splitter.

(Classic)



We express the incident intensity as $I_{\text{in}} = I_0 + \delta I$.

Subsequently, the output intensities are:

$$I_t = \frac{1}{2} I_{\text{in}} = \frac{1}{2} (I_0 + \delta I) \quad I_r = \frac{1}{2} I_{\text{in}} = \frac{1}{2} (I_0 + \delta I)$$

(identical)

Fluctuations at the transmitted and reflected ports are fully correlated.

(Quantum)

We denote: n_{in} as the input photon number,
 n_t as the transmitted photon number,
 n_r as the reflected photon number.

$$\left(\begin{array}{l} n_{\text{in}} = n_t + n_r \\ \bar{n}_t = T \bar{n}_{\text{in}} \\ \bar{n}_r = R \bar{n}_{\text{in}} = (1 - T) \bar{n}_{\text{in}} \end{array} \right)$$

For a given n_{in} , the conditional probability of {transmitted photon number = n_t & reflected photon number = n_r } is

$$P(n_t, n_r | n_{\text{in}}) = \binom{n_{\text{in}}}{n_t} T^{n_t} R^{n_r}$$

(ref.)

(trans.)

of patterns choosing n_1 out of n_0

$$\left(\begin{array}{l} T: \text{transmittance} \\ R = 1 - T: \text{reflectance} \end{array} \right)$$

The probability of coherent light having n_{in} photons is

$$P(n_{\text{in}}) = \frac{\bar{n}_{\text{in}}^{n_{\text{in}}}}{n_{\text{in}}!} e^{-\bar{n}_{\text{in}}}$$

Therefore,

$$\begin{aligned} P(n_t, n_r) &= P(n_t, n_r | n_{\text{in}}) P(n_{\text{in}}) = \frac{\bar{n}_{\text{in}}^{n_{\text{in}}}}{n_{\text{in}}!} e^{-\bar{n}_{\text{in}}} \binom{n_{\text{in}}}{n_t} T^{n_t} R^{n_r} = \frac{\bar{n}_{\text{in}}^{n_t + n_r}}{n_{\text{in}}!} e^{-(\bar{n}_t + \bar{n}_r)} \frac{n_{\text{in}}!}{(n_{\text{in}} - n_t)! n_t!} T^{n_t} R^{n_r} \\ &= e^{-\bar{n}_t} e^{-\bar{n}_r} \frac{(T \bar{n}_{\text{in}})^{n_t} (R \bar{n}_{\text{in}})^{n_r}}{n_t! n_r!} = \frac{(\bar{n}_t)^{n_t}}{n_t!} e^{-\bar{n}_t} \cdot \frac{(\bar{n}_r)^{n_r}}{n_r!} e^{-\bar{n}_r} = P(n_t) P(n_r) \end{aligned}$$

The fluctuation at the transmitted port has nothing to do with that at the reflected port, and vice versa.



The fluctuations of the transmitted and the reflected lights are independent with each other.

Lightwave is assumed to be transmitted through a lossy medium.



(Classic)

We express the incident amplitude as $E_{\text{in}} = E_0 + \delta E$.

Subsequently, the output amplitude is:

$$E_{\text{out}} = \sqrt{T}E_0 + \sqrt{T}\delta E$$

Then, the output intensity is:

$$\begin{aligned} I_{\text{out}} &= |E_{\text{out}}|^2 = |\sqrt{T}E_0 + \sqrt{T}\delta E|^2 && (E_0 \gg \delta E) \\ &= T |E_0|^2 + T |\delta E|^2 + 2T \text{Re}[E_0 \cdot \delta E^*] \approx T |E_0|^2 + 2T \text{Re}[E_0 \cdot \delta E^*] \\ &&& \text{(fluctuation)} \end{aligned}$$

The variance of the fluctuation is:

$$\sigma_c^2 = 4T^2 \langle \text{Re}[E_0 \cdot \delta E^*] \rangle \quad (\text{proportional to the square of the transmittance})$$

(Quantum)

The photon number variance of a coherent state is proportional to the mean photon number. A coherent state propagating through a lossy medium is still a coherent state.



$$\sigma_q^2 \propto \bar{I}_{\text{out}} = T \bar{E}_{\text{in}}$$

(proportional to the transmittance)

The dependency on the transmittance is different in quantum and classical noises.