

Using the formulas introduced in the previous chapter, this chapter describes lightwave in terms of quantum mechanics.

First, the general scheme how to write a physical quantity operator is introduced.

Next, a harmonics oscillator is quantized according to this method.

Subsequently, light is quantized, following the way of quantizing a harmonic oscillator.

Canonical quantization (the procedure of transformation from a classical to a quantum worlds)

In classical mechanics, the position x and the momentum p are the two fundamental physical quantities for expressing physical systems. An arbitrary physical quantity A can be expressed by x and p :

$$A = A(x, p)$$

A typical example is the energy of a physical system, H .

$$H = H(x, p)$$

The canonical quantization is a method of quantizing physical quantities as;

First, a physical quantity is expressed by $\{x, p\}$ in a classical way,

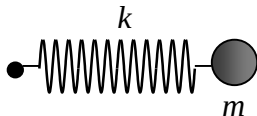
and then $\{x, p\}$ are replaced by operators $\{\hat{x}, \hat{p}\}$ that satisfy

$$[\hat{x}, \hat{p}] = i\hbar \quad \text{(This relationship is presented in the previous chapter.)}$$

Harmonic oscillator

A harmonics oscillator is a physical system that moves in a sinusoidal manner, which can be quantized as follows.

We consider a spring oscillation as an example of a harmonic oscillator.



$$\left(\begin{array}{l} x: \text{position} \\ m: \text{mass} \\ k: \text{spring const.} \end{array} \right)$$

$$m \frac{d^2 x}{dt^2} = -kx \rightarrow x \propto \cos \left[\left(\sqrt{\frac{k}{m}} \right) t \right]$$

$$\text{Sinusoidal oscillation with angular frequency } \omega = \sqrt{\frac{k}{m}}$$

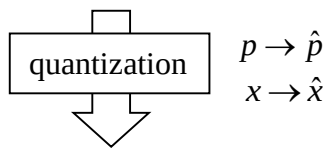
The classical energy of this system is:

$$\begin{aligned} \text{- kinetic energy: } & \frac{1}{2} m v^2 = \frac{p^2}{2m} \\ \text{- potential energy: } & \int_0^x k x' dx' = \frac{k x^2}{2} = \frac{m \omega^2 x^2}{2} \end{aligned}$$

$$\left(\begin{array}{l} v: \text{velocity} \\ p = mv: \text{momentum} \\ \frac{k}{m} = \omega^2 \end{array} \right)$$



$$\text{The total energy is } \frac{p^2}{2m} + \frac{m \omega^2 x^2}{2}$$



$$\begin{aligned} p &\rightarrow \hat{p} \\ x &\rightarrow \hat{x} \end{aligned}$$

The energy operator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$

Eigenvalues of this energy operator, i.e., possible energy values, is derived as follows.

First, the following operators are introduced for the derivation.

$$\left\{ \begin{array}{l} \hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) \\ \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p}) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \end{array} \right.$$

($\hat{a}$ and $\hat{a}^\dagger$ are hermite-conjugated)

($\hat{a} \neq \hat{a}^\dagger$, thus $\hat{a}$ and $\hat{a}^\dagger$ are not Hermitian operators $\rightarrow$ not observable)

The energy operator is expressed by $\{\hat{a}, \hat{a}^\dagger\}$ as

$$\begin{aligned} \hat{H} &= -\frac{1}{2m} \cdot \frac{m\hbar\omega}{2}(\hat{a}^\dagger - \hat{a})(\hat{a}^\dagger - \hat{a}) + \frac{m\omega^2}{2} \cdot \frac{\hbar}{2m\omega}(\hat{a}^\dagger + \hat{a})(\hat{a}^\dagger + \hat{a}) \\ &= -\frac{\hbar\omega}{4}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}) + \frac{\hbar\omega}{4}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}) \\ &= \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}) = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a}) \\ &= \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} \right) \left(\begin{array}{l} [\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \\ = \frac{1}{2m\hbar\omega} \{ (m\omega\hat{x} + i\hat{p})(m\omega\hat{x} - i\hat{p}) - (m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) \} \\ = \frac{1}{2m\hbar\omega} (-i)2m\omega(\hat{x}\hat{p} - \hat{p}\hat{x}) = -\frac{i}{\hbar}[\hat{x}, \hat{p}] = 1 \\ \text{uncertainty relationship} \end{array} \right) \end{aligned}$$

Here, we introduce an operator defined as

$$\hat{n} \equiv \hat{a}^\dagger \hat{a}$$

$$\left[\hat{n}^\dagger \equiv (\hat{a}^\dagger \hat{a})^\dagger = \hat{a}^\dagger \hat{a} = \hat{n} \right]$$

$\hat{n}$ is an Hermitian operator.

Thus, an arbitrary quantum state can be expressed by a linear combination of eigenstates of $\hat{n}$.

Using $\hat{n}$, the energy operator is rewritten as

$$\hat{H} = \hbar\omega \left(\hat{n} + \frac{1}{2} \right)$$

This expression indicates that it is good enough to derive eigenvalues of $\hat{n}$ for obtaining energy eigenvalues.

Thus, we will discuss eigenvalue/eigenstate of $\hat{n}$ in the following.

$$\hat{n} |n\rangle = n |n\rangle$$

$$\left(\begin{array}{l} n: \text{eigenvalue} \\ |n\rangle: \text{eigenstate} \end{array} \right)$$

The inner product with $|n\rangle$

$$\left. \begin{array}{l} \text{left: } \langle n | \hat{n} | n \rangle = \langle n | \hat{a}^\dagger \hat{a} | n \rangle = \|\hat{a} | n \rangle\|^2 \\ \text{right: } \langle n | n | n \rangle = n \langle n | n \rangle = n \end{array} \right\} \longrightarrow n = \|\hat{a} | n \rangle\|^2 \geq 0 \quad \text{non-negative}$$

We operate $\hat{a}$ onto the eigenvalue/eigenstate equation.

$$\text{left: } \hat{a}\hat{n}|n\rangle = \hat{a}\hat{a}^\dagger\hat{a}|n\rangle = (\hat{a}\hat{a}^\dagger)\hat{a}|n\rangle = (\hat{a}^\dagger\hat{a}+1)\hat{a}|n\rangle = (\hat{n}+1)\hat{a}|n\rangle$$

$$\text{right: } \hat{n}\hat{a}|n\rangle$$



$$\hat{n}\hat{a}|n\rangle = \hat{a}|n\rangle = \hat{n}\hat{a}|n\rangle$$

$$\hat{n}\hat{a}|n\rangle = (n-1)\hat{a}|n\rangle \rightarrow \hat{a}|n\rangle \text{ is an eigenstate of } \hat{n}, \text{ with an eigenvalue of } (n-1).$$

From the definition,

an eigenstate with an eigenvalue of $(n-1)$ is $|n-1\rangle$

$$\hat{a}|n\rangle = c_n|n-1\rangle \quad \left[c_n: \text{proportional const.} \right]$$

(inner product with itself)

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n-1|c_n^*c_n|n-1\rangle$$

$$\langle n|\hat{n}|n\rangle = |c_n|^2 \langle n-1|n-1\rangle$$

$$n = |c_n|^2$$

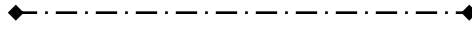
$$c_n = \sqrt{n}$$

(Phase can be 0 without loss of generality.)

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$\hat{a}$ is an operator that decreases n by 1

annihilation operator



From the above,

$$\left\{ \begin{array}{l} \text{Eigenvalue of } \hat{n}, n, \text{ is non-negative.} \\ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \end{array} \right. \Rightarrow \boxed{n \text{ is a natural number.}}$$

Consecutively operate $\hat{a}$ onto $|n\rangle$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \rightarrow \hat{a}\sqrt{n}|n-1\rangle = \sqrt{n(n-1)}|n-2\rangle \rightarrow \dots \rightarrow \sqrt{\frac{n!}{m!}}|n-m\rangle$$

If n is not a natural number $\rightarrow \hat{a}|0.5\rangle = \sqrt{0.5}|(-0.5)\rangle$ n becomes negative. NG

If n is a natural number $\rightarrow \hat{a}|0\rangle = 0$



n is a natural number for n to be non-negative

Similarly for $\hat{a}^\dagger$

$$\hat{n}|n\rangle = n|n\rangle$$

$$\text{left: } \hat{a}^\dagger\hat{n}|n\rangle = \hat{a}^\dagger\hat{a}^\dagger\hat{a}|n\rangle = \hat{a}^\dagger(\hat{a}^\dagger\hat{a})|n\rangle = \hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)|n\rangle = (\hat{n}\hat{a}^\dagger - \hat{a}^\dagger)|n\rangle$$

$$\text{right: } \hat{n}\hat{a}^\dagger|n\rangle$$



$$(\hat{n}\hat{a}^\dagger - \hat{a}^\dagger)|n\rangle = n\hat{a}^\dagger|n\rangle$$

$$\hat{n}\hat{a}^\dagger|n\rangle = (n+1)\hat{a}^\dagger|n\rangle \rightarrow \hat{a}^\dagger|n\rangle \text{ is an eigenstate of } \hat{n}, \text{ with an eigenvalue of } (n+1).$$

$$\begin{aligned} & \downarrow \\ & \hat{a}^\dagger|n\rangle = d_n|n+1\rangle \\ & \downarrow \\ & \langle n|\hat{a}\hat{a}^\dagger|n\rangle = |d_n|^2 \\ & \langle n|(\hat{a}^\dagger\hat{a}+1)|n\rangle = |d_n|^2 \\ & \langle n|(\hat{n}+1)|n\rangle = |d_n|^2 \\ & n+1 = |d_n|^2 \\ & \downarrow \\ & d_n = \sqrt{n+1} \end{aligned}$$

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$\left[\hat{a}, \hat{a}^\dagger \right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$

$\hat{a}^\dagger$ is an operator that increases n by 1

creation operator

Utilizing the above results, eigenvalues of the energy operator are obtained as;

$$\hat{H} = \hbar\omega \left(\hat{n} + \frac{1}{2} \right)$$

$$\hat{H}|n\rangle = \hbar\omega \left(\hat{n} + \frac{1}{2} \right) |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle$$

eigenvalue $E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad (n = 0, 1, 2, 3, \dots)$

Energy eigenvalue of a harmonic oscillator.

discrete in the quantum world

Energy of electromagnetic wave

Now, let us describe an electromagnetic wave quantum-mechanically, based on the previous section. First, we will see that the structure of the expression of the energy for an electromagnetic wave is similar to that for a harmonic oscillator.

Subsequently, electromagnetic wave will be quantized, following the quantization of a harmonic oscillator.

An electromagnetic wave is classically expressed as

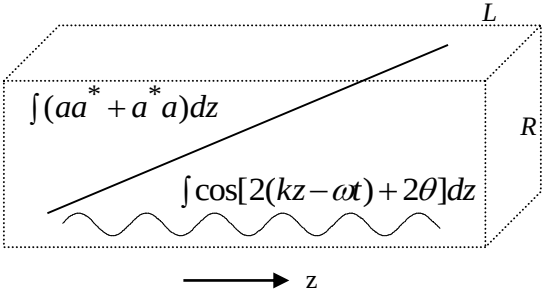
$$\begin{cases} E_x(z,t) = E_0 e^{i(kz-\omega t)} + E_0^* e^{-i(kz-\omega t)} \\ B_y(z,t) = B_0 e^{i(kz-\omega t)} + B_0^* e^{-i(kz-\omega t)} \end{cases} \quad \text{where} \quad B_0 = \frac{E_0}{c}$$

(rewrite)

$$\begin{cases} E_x(z,t) = A \{ i a e^{i(kz-\omega t)} - i a^* e^{-i(kz-\omega t)} \} \\ B_y(z,t) = \frac{A}{c} \{ i a e^{i(kz-\omega t)} - i a^* e^{-i(kz-\omega t)} \} \end{cases}$$

$$\left(\begin{array}{l} a: \text{normalized amplitude} \\ A = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \quad \text{constant for } a \text{ to be non-dimensional.} \end{array} \right)$$

The classical energy of an electromagnetic wave is

$$\begin{aligned}
 H &= \frac{1}{2} \int_V (\epsilon_0 E_x^2 + \mu_0 H_y^2) dv \\
 &= \frac{\epsilon_0 R}{2} \int_0^L (E_x^2 + \frac{1}{\epsilon_0 \mu_0} B_y^2) dz \quad \left(\begin{array}{l} R: \text{cross section area in the x-y plain} \\ L: \text{length along the z direction} \end{array} \right) \\
 &= \epsilon_0 R \int_0^L E_x^2 dz \\
 &= -A^2 \epsilon_0 R \int_0^L \{ a^2 e^{2i(kz - \omega t)} + (a^*)^2 e^{-2i(kz - \omega t)} - aa^* - a^* a \} dz \\
 &= -A^2 \epsilon_0 R \int_0^L \underbrace{\{ 2|a|^2 \cos[2(kz - \omega t) + 2\theta] \}}_{\text{oscillating around 0}} \underbrace{-aa^* - a^* a}_{\text{linearly increasing}} dz \\
 &\approx A^2 \epsilon_0 R \int_0^L \{ aa^* + a^* a \} dz \\
 &= A^2 \epsilon_0 RL (aa^* + a^* a) \\
 &= \frac{\hbar \omega}{2} (aa^* + a^* a)
 \end{aligned}$$


Here, we decompose a into the real part q and the imaginary part p as

$$a = \frac{1}{\sqrt{2\hbar\omega}} (\omega q + ip) \quad a^* = \frac{1}{\sqrt{2\hbar\omega}} (\omega q - ip)$$

Substitute them into energy H .

$$H = \frac{1}{2} (p^2 + \omega^2 q^2) \quad \leftarrow$$

On the other hand, the energy of a harmonic oscillator is classically written as

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \quad \leftarrow$$

(p : momentum, x : position, m : mass)



An electromagnetic wave can be quantized similarly to a harmonic oscillator.

The same structure with $m = 1$.

Comparison of the energy expressions of an electromagnetic wave and a harmonic oscillator

EM wave		harmonic
p	$\Leftrightarrow$	p
q	$\Leftrightarrow$	x

From this relationship, $\{p, q\}$ of electromagnetic wave are quantized as $\{p, x\}$ of an harmonic oscillator as:

$$\begin{cases} p \rightarrow \hat{p} \\ q \rightarrow \hat{q} \end{cases}$$

with

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar$$

According to the quantization of $\{p, q\}$, $\{a, a^*\}$ are also quantized as:

$$\begin{cases} a \rightarrow \hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) & \text{annihilation operator satisfying } \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \\ a^* \rightarrow \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}) & \text{creation operator satisfying } \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \end{cases}$$

Let us confirm the commutation relationship.

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \frac{1}{2\hbar\omega} \{(\omega\hat{q} + i\hat{p})(\omega\hat{q} - i\hat{p}) - (\omega\hat{q} - i\hat{p})(\omega\hat{q} + i\hat{p})\} \\ &= \frac{1}{2\hbar\omega} (\omega^2\hat{q}^2 - i\omega\hat{q}\hat{p} + i\omega\hat{p}\hat{q} + \hat{p}^2 - \omega^2\hat{q}^2 - i\omega\hat{q}\hat{p} + i\omega\hat{p}\hat{q} - \hat{p}^2) \\ &= -\frac{i}{\hbar} [\hat{q}, \hat{p}] = 1 \end{aligned} \quad \boxed{[\hat{a}, \hat{a}^\dagger] = 1}$$

According to the above quantization, the energy of electromagnetic wave is quantized as:

$$\begin{aligned} H = \frac{1}{2}(p^2 + \omega^2 q^2) &\longrightarrow \hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2) = \frac{1}{2} \left[-\frac{\hbar\omega}{2} (\hat{a} - \hat{a}^\dagger)^2 + \omega^2 \frac{\hbar}{2\omega} (\hat{a} + \hat{a}^\dagger)^2 \right] \\ &= \frac{\hbar\omega}{2} (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\ &= \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} \right) \end{aligned}$$

Similarly to a harmonic oscillator, we introduce an operator defined as:

$$\hat{n} = \hat{a}^\dagger\hat{a}$$

which is an Hermitian operator, and has its eigenvalue/eigenstate as:

$$\hat{n}|n\rangle = n|n\rangle$$

- n is a measured value of physical quantity $\hat{n}$.
- n is a natural number ($n = 0, 1, 2, 3, \dots$)

Using $\hat{n}$, the energy operator is rewritten as

$$\hat{H} = \hbar\omega \left(\hat{n} + \frac{1}{2} \right)$$

Subsequently, the energy operator of an electromagnetic wave has eigenvalue/eigenstate as

$$\hat{H}|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle$$

This equation indicates that energy values of an electromagnetic wave are discrete, and is written as

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad (n = 0, 1, 2, 3, \dots)$$

Here, we should recall that the energy of lightwave has a minimum unit (i.e., *photon*) and thus is discrete.

Also, the energy of one photon is $h\nu$ or $\hbar\omega$.

The above eigenvalues of the energy operator indicate these postulates.

Eigenstate $|n\rangle$ represents the state of light whose energy value is $\hbar\omega(n + \frac{1}{2})$

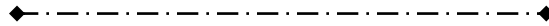
or whose photon number is n .

This state is called “photon number state” or “Fock state”.

Note in the expression of eigenvalues that light has an energy of $\frac{\hbar\omega}{2}$ even when it has no photon.

This is called “zero-point energy” originating from the vacuum fluctuation or the quantum fluctuation.

The details will appear in the next chapter.



Finally, an electromagnetic wave is quantized, based on the above discussion.

$$\hat{E}(z, t) = i \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \{ \hat{a} e^{i(kz - \omega t)} - \hat{a}^\dagger e^{-i(kz - \omega t)} \}$$

This operator is an Hermitian operator.

$$\hat{E}^\dagger = -i \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \{ \hat{a}^\dagger e^{-i(kz - \omega t)} - \hat{a} e^{i(kz - \omega t)} \} = \hat{E}$$

This expression will be used in the following chapters.